



CONDITIONAL RISK MEASURES FOR THE MULTISTAGE WEALTH
ALLOCATION PROBLEM

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Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Engenharia de Produção, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Engenharia de Produção.

Orientador: Juan Pablo Cahahuanca Luna

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MEDIDAS DE RISCO CONDICIONAIS PARA O PROBLEMA DE ALOCAÇÃO DE RIQUEZA MULTISTÁGIO

Fernando Queiroz de Lira Alexandrino

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Orientador: Juan Pablo Cajahuanca Luna

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Este trabalho aborda o modelo de alocação de riqueza. O problema tem duas questões principais. A primeira é como lidar com a incerteza nos preços das ações. Como a decisão está em um contexto multiestágio, uma medida de risco condicional deve ser adotada. A segunda questão é relacionada à resolução do problema de otimização. A fim de evitar a maldição da dimensionalidade e permitir o uso de técnicas eficientes, como o *Stochastic Dual Dynamic Programming* – SDDP, é necessário que a formulação tenha uma estrutura especial. Dependendo da medida de risco, esta estrutura pode se tornar inviável. O objetivo deste trabalho é apresentar formulações para o problema de alocação de riqueza multiestágio usando medidas de risco condicionais e garantir tal estrutura. Dois modelos foram propostos e implementados em um estudo de caso. Diferentes casos foram investigados alterando as estratégias de operação, os níveis de risco/retorno e o uso de medidas com e sem a propriedade da consistência de tempo. A metodologia foi dividida em três etapas: composição do portfólio usando as ações do índice IBrX-50; geração da árvore de cenários; e otimização do problema com o SDDP. Os resultados mostraram que foi possível obter altos retornos e manter as perdas abaixo de um limite.

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CONDITIONAL RISK MEASURES FOR THE MULTISTAGE WEALTH ALLOCATION PROBLEM

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This work addresses the wealth allocation model. The problem has two main issues. The first is how to deal with uncertainty on the stock prices. As the decision is in a multistage context, a conditional risk measure should be adopted. The second issue is related to solve the optimization problem. In order to avoid the curse of dimensionality and allow the use of efficient techniques, as Stochastic Dual Dynamic Programming – SDDP, it is required that the formulation has a special structure. Depending on the risk measure, this structure may become infeasible. The objective of this work is to present formulations for the multistage wealth allocation problem using conditional risk measures and to ensure such structure. Two models were proposed and implemented in a case study. Different cases were investigated by changing the operation strategies, the risk/profit levels and the use of measures with and without the time consistency property. The methodology was split in three steps: composition of the portfolio using the assets on IBrX-50 index; scenario tree generation; and optimization of the problem with SDDP. The results shows that it is possible to obtain high returns and keep the losses under a threshold.

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Chapter 1

Introduction

Investment on financial market is a complex task. Basically, two important variables should be considered: risk and return. The risk could be defined as the possibility of occurrence of unfavorable and unexpected events, usually leading to negative results, such as capital loss. It is composed of market risk, operational risk, credit risk and others variations. The return is the reward that can be achieved by the agent for assuming risk. Market risk – the main focus of this work – is directly related to the way assets price behave in face of market conditions. In the case of a portfolio, the risk depends on the correlation between the individual assets and can be reduced with capital diversification.

As it is known, the risk-return binomial tend to have positive correlation. So, since high returns are related with high risks, a key issue for the investment success is a good risk management. And to manage it is necessary to measure it. Risk measures synthesize the risk of the investment using a real number and they are a highly studied topic in portfolio optimization. The use of measures to minimize or limit losses forces the policies to diversify the portfolio. Consequently it is possible to reduce investments risks and obtain a higher return. Quantitative risk management began from Markowitz theory [1] and it is the central issue of modern research on finance.

The objective of the wealth allocation problem is to determine a strategy to invest an initial amount of money over a period of time, avoiding losses. As the future is uncertain and many variables are not under control the problem can be solved using scenarios. These scenarios are sequences of states, representing possible

instances or parameters combination that compose a given event, associated with an occurrence probability. In addition, it is reasonable that the choices change over time, defining the stages of the decision. At each stage, the agent needs to make a decision according to information available until that moment and considering the scenarios that have been traced for the future. These features lead to a multistage stochastic programming problem.

1.1 Motivation

In many markets – as it happens in Brazil – a considerable part of investors is not aware of risk measures and stochastic models that can help finance trading decisions and its effect on portfolio diversification. An agent that is proposed to measure and limit risk, but does not use appropriated measures, may be exposed to losses or opportunity costs, because he made his choice considering poor information. Similarly, the use of deterministic models leads to solutions based on a single representation of the problem, which may have low occurrence probability and the decisions taken may not be good for other scenarios that may occur.

The literature has several studies about risk-aversion optimization, which can be focused either on the modeling of the asset price series, on methodologies to adequately measure the risk or on the algorithms for solving the stochastic problem. A few works solve this problem in a single stage using assets from the Brazilian stock exchange [2, 3]. A multistage decision program should consider conditional risk measures and it also requires special techniques to be efficiently solved. Because the curse of dimensionality, the problem easily becomes prohibitive. These techniques are based on Benders decomposition. However, the problem must have a special structure.

One of the first studies of the multistage wealth allocation problem was [4]. This work considers a portfolio with 13 assets, 3 stages and 50 scenarios per stage. More recently, in [5] the problem is solved in 5 stages. The portfolio is composed by 11 assets and the uncertainty is modeled using 1000 scenarios per stage. In [6], a case study considers 4 assets, 15 stages and 50 scenarios at each stage. To overcome the computational effort, in [7] an analytical metric to limit the risk is proposed instead

optimization algorithms. The problem contains 10 assets and the investment period has 50 stages.

Observe that the number of assets is relatively small compared to all the options available to the investor. Considering more assets would rise the computational time, and this is already an issue when solving the problem. In order to provide a more realistic approach of the problem, the models should receive a given risk tolerance level or return ratio and optimize the portfolio using a large set of scenarios and also a large set of assets. This context justify the relevance of this study.

1.2 Objective and Contributions

This work proposes models for the wealth allocation problem using the multi-stage stochastic programming with conditional risk measures. Numerical tests were performed using a case study with a large set of assets from the Brazilian stock exchange. To achieve this objective, it was necessary to perform the following tasks:

1. Perform a revision of stock price models commonly used on finance, discussing their assumptions and applicability;
2. Extend one-period conditional risk measures to the multistage case;
3. Propose different models for the multistage wealth allocation problem;
4. Select a large set of assets from the Brazilian stock exchange to compose the portfolio;
5. Generate the scenario tree to represent the problem uncertainty using historical stock prices;
6. Solve the optimization problem using an efficient technique for large-scale instances; and
7. Analyze the solutions from different points of view, such as the composition of the portfolio, the return on investment and the losses distribution.

The main contribution of this work is the methodology for dealing with the wealth allocation problem. We propose two multistage stochastic programming

based models that considers risk over losses. Whereas our models are rather general, in order to make them feasible for real life applications, we provide some mechanisms for solving numerically large instances of the models. To the best of our knowledge, these large-sized numerical backtests were never performed in the literature.

1.3 Outline

This work is organized in six chapters. The first makes an introduction to the theme. Chapter 2 consists of a literature review on the modeling of assets price through scenario trees, as well as risk measures appropriate to this class of problems. It also presents an algorithm that can solve stochastic multistage problems efficiently. Throughout this chapter references are made to various similar works, both in quantitative finance and other areas. Chapter 3 is dedicated to discuss two formulations for the multistage wealth allocation problem. Chapter 4 details the methodological procedures followed during the problem optimization, whose results are presented in Chapter 5 – the portfolio compositions found in this work are also available in Appendix A. Finally, Chapter 6 gives a few future research suggestions.

Chapter 2

Literature Review

Financial assets involve risk. Recent studies on risk-averse optimization have focused on two important topics: modeling techniques for stock price and measures to avoid losses associated to investment choices. A third topic is related to multistage stochastic programming, that usually involves high computational cost both in terms of the number of scenarios and the use of non-linear elements.

2.1 Scenario Generation

The first topic corresponds to the Scenario Generation (SG). Decision-making in finance occurs in a context of high uncertainty about the future. In many practical applications this uncertainty is addressed through data aggregation, taking for example the average of a given set of observations and adopting it as a prediction. When there is a lot of unknown information the averages of each of them lead to a single representation of the problem making it deterministic, and there are few guarantees that the optimal solution of this configuration is also optimal (or simply good) for all other cases, especially to the one which in fact will happen. SG is a process of creation of finite realizations that describe the vector of random variables of the parameters of the problem accompanied by their respective probabilities of occurrence.

In [8] the desirable properties of SG in finance are found. *Correctness* implies that the set of scenarios is a “correct” representation of the random returns of the assets, but as we do not know the actual distributions, the term “correct” is related

to the model believed to be the most appropriate to approximate the dynamics of prices. For example, several models consider that the volatility of stock prices varies over time, while for others it is constant. Since the random variables are correlated, *consistency* means the values of a given scenario must be correlated with each other. In other words, given a group of assets in the same industry sector it is expected that the returns of these assets has the same behavior in any scenario, be it gain or loss. Finally, *stability* leads to a “stable” decision, that is, the optimal solutions obtained when solving optimization problems with different SGs do not vary significantly in themselves.

Several works have sought to evaluate the quality of the scenario generators (see [9, 10]) but this is a difficult task since the actual values are unknown. One way would be to choose a date prior to the date of the last available information for SG and then compare with the data already known. Once the method is validated, all the information available so far is used to generate the future scenarios. However, the main concern in SG is its capability to support a good decision. Since the scenarios are not a prediction of the future, some are optimistic and other pessimistic, the investment strategy should perform well in any case [8].

One of the most used asset price models is the Geometric Brownian Motion (GBM), which approximates the stock prices – stochastic processes in discrete time – through continuous time stochastic processes. Let be $S_t = (S_t^1, S_t^2, \dots, S_t^n)$ a vector whose each entry represents the price of asset $i = 1, \dots, n$ at time t . These prices may be correlated. It is assumed that the price vector S_t follows a multidimensional GBM, i.e.

$$dS_t = S_t \circ (\mu dt + AdW_t), \quad (2.1)$$

where the operator \circ means the product between two vectors is made termwise, $\mu \in \mathbb{R}^n$ and W_t is described by a multidimensional Wiener Process [11].

The solution to this system of stochastic differential equations is

$$S_t = S_0 \circ \exp \left(\left(\mu - \frac{1}{2} \text{diag}(\Sigma) \right) t + AW_t \right) \quad (2.2)$$

where $\Sigma = AA^\top$ and $\text{diag}(\Sigma)$ is the vector whose entries are the elements of the diagonal of Σ . The price process can be rewritten as

$$\log(S_t) = \log(S_0) + \left(\mu - \frac{1}{2} \text{diag}(\Sigma) \right) t + AW_t. \quad (2.3)$$

Note that for $\Delta t > 0$

$$\log(S_{t+\Delta t}) = \log(S_t) + \left(\mu - \frac{1}{2}\text{diag}(\Sigma)\right)\Delta t + A(W_{t+\Delta t} - W_t). \quad (2.4)$$

This expression can be used for simulating the process S_t . For this, note that for a sample $S_0, S_{\Delta t}, S_{2\Delta t}, \dots, S_{T\Delta t}$ and defining the returns by

$$r_k = \log(S_{k\Delta t}) - \log(S_{(k-1)\Delta t}), \quad k = 1, 2, \dots, T \quad (2.5)$$

it is possible to have, because of the properties the brownian motion W_t , that r_1, r_2, \dots, r_T are i.i.d. with joint normal distribution $\mathcal{N}(\mu', \Sigma')$, where $\mu' = \Delta t(\mu - \frac{1}{2}\text{diag}(\Sigma))$ and $\Sigma' = \Delta t\Sigma$. For estimating the parameters of this distribution the maximum likelihood criterion can be used, leading to

$$\mu' = \frac{1}{T} \sum_{k=1}^T r_k \quad \text{and} \quad \Sigma' = \frac{1}{T} \sum_{k=1}^T (r_k - \mu')(r_k - \mu')^\top. \quad (2.6)$$

Finally, to simulate a new samples of prices, it is enough to simulate a sample of log-returns r_k and take into account that

$$S_{k\Delta t} = S_{(k-1)\Delta t} \exp(r_k), \quad k = 1, 2, \dots, T. \quad (2.7)$$

Note that, for generating a sample of log-returns r_k knowing μ' and Σ' , it is possible to generate a sample N_k of $\mathcal{N}(0, I_n)$ and make $r_k = \mu' + \sqrt{\Delta t}AN_k$. The matrix A can be computed from Σ' by performing a Cholesky factorization.

Although GBM has been used in many works (e.g. [12, 13, 7]), it considers that price volatility is constant over time, which is a reasonable assumption when dealing with a short-term period. However, increasing the time horizon, the financial market presents a cyclical behavior, alternating periods of low and high volatility. This phenomenon is known as ‘‘volatility clustering’’ and implies that the volatility perturbations of returns will have an influence on the expected volatilities for the future. In other words, the information that comes to the market is correlated in time [14].

Consider then that the sigma-algebra \mathcal{F}_t represents the information available in time t . It is possible to write $r_t = \bar{r}_t + \varepsilon_t$, where \bar{r}_t is the expected value of r_t and ε_t is the error process. GARCH models – from Generalized Autoregressive Conditional Heteroskedasticity [15] – are an approach to evaluate the volatility of returns. The

GARCH(η, ν) process is

$$\begin{aligned} \varepsilon_t | \mathcal{F}_{t-1} &\sim N(0, \sigma_t^2) \\ \sigma_t^2 &= \pi_0 + \pi_1 \varepsilon_{t-1}^2 + \cdots + \pi_\eta \varepsilon_{t-\eta}^2 + \psi_1 \sigma_{t-1}^2 + \cdots + \psi_\nu \sigma_{t-\nu}^2 \end{aligned} \tag{2.8}$$

where $\eta \geq 0$, $\nu > 0$, $\gamma_0 > 0$, $\pi_\eta \geq 0$ and $\psi_\nu \geq 0$. Note that the variance depends on its own previous values and on the previous error processes. For SG purposes the maximum likelihood criterion allows to estimate the parameters π_η and ψ_ν , that are used to compute σ_t^2 and it is possible to generate returns samples r_k recursively.

GARCH models have been extended in the most distinct directions. Surveys on the multivariate approaches regarding the time evolution of the prices volatility can be seen in [16, 17]. Several works have sophisticated the original model considering asymmetric volatility [18, 19, 20], that is, negative news tend to generate a greater and faster disturbance in volatility when compared to positive news. In [21, 22] the modeling is turned to the standard deviation instead of the variance. A method to evaluate the quality of scenarios generated by different models is found in [23].

Recent studies on SG in finance have used regime-switching approaches, especially Hidden Markov Models (HMM). Basically, these models consider that there is another “hidden” stochastic process behind the stochastic process of returns representing market scheme changes over time. At each instant t the market is in one of several states and S_t has a different distribution for each one. In this perspective, single models (like GBM and GARCH) are valid only for short periods of time while HMM are suitable for long-term modeling.

Another advantage is that regime-switching contributes to the representation of extreme events which occur with low probability and are the result of given conditions of the market. It is especially useful to consider these events when solving optimization problems with risk aversion, because it is desirable to have a decision that is good in any scenario that may occur. The optimal solution minimizes losses considering the whole set of scenarios, even if this “penalizes” the objective function of those most optimistic; after all, the scenario that will actually occur is unknown. Papers involving HMM in finance can be found in [10, 24, 25]. In addition to GBM, GARCH and its extensions and HMM, the literature also has other techniques for approaching stock prices, such as stochastic volatility models and grey theory (see for example [26, 27]).

Finally, it is possible to highlight two other issues related to SG. The first one is scenario tree reduction. In multistage stochastic programs, it is convenient to consider that the sigma-algebras \mathcal{F}_t are all finite for $t \in \mathcal{D} = \{1, 2, \dots, T\}$. This process determines a filtration $\mathcal{F}_{\mathcal{D}} = \{\mathcal{F}_t\}_{t \in \mathcal{D}}$ that can be modeled as a tree. For any time t , since \mathcal{F}_t is finite, it is determined by a partition of probability space. This partition is denoted by Ω_t . Note that when representing the filtration as a tree, the set Ω_t is identified with the set of nodes at level t . Also, for each element $a \in \Omega_t$ and $d > t$ it is possible to define the set $\mathcal{C}_a^d = \{b \in \Omega_d : b \subset a\}$. In the language of trees, \mathcal{C}_a^d represents the set of all nodes at level d that are descendants of a . In particular the set \mathcal{C}_a^{t+1} of all children of node a is denoted by \mathcal{C}_a . Note that if a random variable Z is \mathcal{F}_t -measurable, then it is constant on each element of Ω_t , and so, it can be identified by a vector $(Z_a)_{a \in \Omega_t} \in \mathbb{R}^{|\Omega_t|}$.

When the tree becomes reasonably large it may be necessary to employ some algorithm that performs scenario reduction in order to reduce the computational effort of the optimization problem. For each level t of the tree, it is found an optimal set Ω_t^* containing the nodes that best represent the distribution at this level, minimizing a given measure δ (see [28] to correctly select δ). In general, these algorithms have an outer iteration that adds elements to Ω_t^* , followed by an inner iteration that calculates the conditional probabilities. The first step consists of a combinatorial problem of type $|\Omega_t^*|$ -median, which is \mathcal{NP} -hard. For this reason it is common to use heuristic methods. For scenario tree reduction techniques and numerical examples, see [29].

The second issue to be highlighted is *ex post* evaluation. When considering SG as input for stochastic programs we are using only one of the functions of the scenarios (*ex ante*). Another direction that can be explored is to evaluate the quality of the decisions obtained after solving the optimization problem by simulating the future. Many works on stochastic programming with risk aversion perform a backtesting step where the optimal solutions are confronted with historical data (or it is waiting some time to collect new observations) in order to verify if the real losses have remained under the adopted limits. If the results are satisfactory, a good performance is also expected in the future. However, this approach validates the results of a single realization of the random parameters (the one that actually occurred).

Then SG can be used to obtain a large set of scenarios for the future and analyze the performance of the decision in these scenarios, using for example risk measures, statistical measures – of centrality and dispersion – or Sharpe ratio and Jensen’s alpha, for example, in a context of finance. A discussion on *ex post* evaluation can be seen in [8].

2.2 Conditional Risk Measures

The risk of an investment was always a relevant aspect. Quantitative risk management began from Markowitz theory [1] and it is the central issue of modern research on finance. A significant portion of the risk is associated with random causes that could be eliminated with portfolio diversification. The use of measures to minimize or limit losses forces the policies to distribute the wealth among the assets. Risk measures are intended to synthesize the risk of a financial position using a real number.

The Value-at-Risk (VaR) is the most widely used and known measure and reports the worst loss that can occur in a given time horizon for a given risk tolerance level. However, if on one side are the popularity and ease of implementation and verification of VaR, on the other side are two negative aspects to its use. The first is that it does not model the principle of financial diversification, i.e. a diversified portfolio can raise the risk rather than reduce it. This occurs, for example, when the dependence on the marginal distributions of the assets is highly asymmetric. The second disadvantage is the lack of information about the magnitude of the loss is when it exceeds the VaR. A more sophisticated risk measure is the Average Value-at-Risk (AVaR), which corrects the failures of its predecessor and can be formulated as a linear programming problem [30]. For these reasons, other measures were developed from AVaR. In [31] a chronological evolution of them is presented. A comprehensive and modern survey of financial risk measures as well as their extensions can be found in [32].

In [3] a model for portfolio selection is presented which is based on the risk analysis of financial assets. The scenario tree of the returns was generated using variations of the GARCH model and VaR and AVaR measures were used to model

the risk. Both are sensitive to the sample size and are appropriate for studies where large samples are available or when sample size is controlled. In [33] a portfolio optimization problem is solved in a financial institution focused on the management of pension funds with considerable volumes of capital. For this, a model that contemplates the benefits generated by the inclusion of derivatives and AVaR was formulated. The study uses assets price simulation and option pricing with portfolio optimization, aiming at minimizing the AVaR.

Risk measures are also used in areas other than finance. The work [34] presents studies aimed at optimizing the expected return of a portfolio composed by investment assets of a company in the energy segment, which is responsible for the acquisition, sale and transportation of pipelines. The objective of the work was to develop a model that supported the portfolio selection with the presence of some degree of irreversibility, in order to meet given criteria of optimality and operational and financial constraints. The risk was treated using AVaR. The author uses techniques of decomposition by cutting planes and Lagrangian relaxation. It was possible to obtain results with relatively small gap and satisfactory computational effort.

In a context of multistage decision problem, a conditional risk measure should be considered. Let be \mathcal{M}_t the space of random variables being considered that are \mathcal{F}_t -measurable and $Z_t \in \mathcal{M}_t$. A conditional coherent risk measure is a mapping $\rho(Z_{t+1}) : \mathcal{M}_{t+1} \rightarrow \mathcal{M}_t$ such that it satisfies the following properties. Assume that Z_t represents a loss distribution.

(i) *Positive homogeneity:* $\rho(\gamma Z_{t+1}) = \gamma \rho(Z_{t+1}), \quad \gamma \geq 0$

(ii) *Monotonicity:* $\rho(Z_{t+1}) \geq \rho(Z'_{t+1}), \quad \text{if } Z_{t+1} \geq Z'_{t+1}$

(iii) *Convexity:* $\rho(\gamma Z_{t+1} + (1 - \gamma)Z'_{t+1}) \leq \gamma \rho(Z_{t+1}) + (1 - \gamma)\rho(Z'_{t+1}), \quad \gamma \in [0, 1]$

(iv) *Translation equivariance:* $\rho(Z_t + Z_{t+1}) = Z_t + \rho(Z_{t+1})$

Positive homogeneity indicates that the risk of a financial position is proportional to its size. The axiom of monotonicity shows that positions that lead to larger losses (e.g. $Z_{t+1} > Z'_{t+1}$) also have higher risks. Convexity models the principle of diversification and, as a product, it allows us to use convex theory to facilitate

our analysis. The translation equivariance implies that by adding (subtracting) a given value to the variable, as it models a loss situation, the risk measure increases (decreases) by the same amount – note that $Z_t \in \mathcal{M}_t$.

An example is the Conditional Average Value-at-Risk $CAVaR_\alpha(Z_{t+1}|\mathcal{F}_t) : \mathcal{M}_{t+1} \rightarrow \mathcal{M}_t$ with risk tolerance level $0 < \alpha < 1$, defined by [35, p. 315]

$$CAVaR_\alpha(Z_{t+1}|\mathcal{F}_t)(\omega_t) = \min_{u \in \mathcal{M}_t} \left\{ u(\omega_t) + \frac{1}{\alpha} \mathbb{E}[[Z_{t+1} - u]^+ | \mathcal{F}_t](\omega_t) \right\} \quad (2.9)$$

where ω_t is a particular outcome from the sample space Ω_t .

Once a given measure ρ is chosen the risk can be modeled in different ways. The first one is adopting the *sum of stage-wise independent risks*. In this case,

$$\rho(Z_1, \dots, Z_T) = Z_1 + \rho_2(Z_2) + \dots + \rho_T(Z_T). \quad (2.10)$$

This approach is quite intuitive: the risk is measured at each date separately. However, minimizing the function (2.10) can lead to inconsistent policies.

Time consistency is a desirable property in risk measures and a concept still under discussion in the literature. According to [36] a policy is (time) consistent when the decisions made in t are in agreement with the planning made in $t - 1$ for the scenario that actually occurred. In other words, assume the problem is solved once and there is a solution for each node of the scenario tree. When solving the problem again at a given future stage and considering the new information available until this new date, the same solutions should be found. These authors prove that the measure (2.10) is inconsistent when $\rho_t = CAVaR_\alpha$ and α moves away from zero – that is, when the solution is no longer driven by the worst case path.

Another approach is to consider a *nested multistage risk measure* $\tilde{\rho} : \mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_T \rightarrow \mathbb{R}$, defined by

$$\tilde{\rho}(Z_1, \dots, Z_T) = \rho_1 \circ \rho_2 \circ \dots \circ \rho_T(\Sigma_t Z_t). \quad (2.11)$$

This measure will always lead to consistent policies according to the concept adopted in [36]. In fact, nested risk measures imply time consistency [37]. The problem can be written recursively by stage through Bellman equations and optimized using dynamic programming algorithms, but obtaining upper limits for the objective function is not a trivial task (see [38, 6]). In [35, p. 326] a convex combination between the expectation of the portfolio's value and CAVaR in

the last period is proposed. A similar approach is seen in [5], where a function $\rho_t(Z_t) = (1 - \Lambda_t)\mathbb{E}[Z_t|\mathcal{F}_{t-1}] + \Lambda_t\text{CAVaR}_\alpha(Z_t|\mathcal{F}_{t-1})$, with $\Lambda_t \in [0, 1]$, is used to implement a risk aversion policy. Other formulations for nested measures in multistage stochastic programs can be found in [39, 40].

The last way to model risk is by adopting a *multi-period composite measure*, i.e.

$$\hat{\rho}(Z_1, \dots, Z_T) = Z_1 + \rho_2(Z_2) + \mathbb{E}_{\omega_{[2]}}[\rho_3^{\omega_{[2]}}(Z_3)] + \dots + \mathbb{E}_{\omega_{[T-1]}}[\rho_T^{\omega_{[T-1]}}(Z_T)], \quad (2.12)$$

where $\omega_{[t]}$ denotes the history until t . This is an intermediate approach between separated and nested cases. The index in $\mathbb{E}_{\omega_{[t-1]}}$ and $\rho_t^{\omega_{[t-1]}}$ indicates that the expectation and the measure are with respect to the history $\omega_{[t-1]}$. In [36] the authors call the function $\hat{\rho}$ expected conditional risk measure and prove that it is consistent. One advantage of this approach is a better understanding of how risk is being treated, which is hard to note when considering nested measures. In addition, it is possible to implement existing neutral-risk algorithms directly to optimize the function $\hat{\rho}$. Applications using composite measures, especially in the particular case when $\rho_t = \text{CAVaR}_\alpha$, can be seen in [36, 41, 42].

Time consistency was also presented by other works. According to [35, p. 321] it occurs when at each stage of the problem the optimal decisions should not depend on scenarios which will not happen. For [43], a policy is consistent when decisions planned for the future stages will actually be implemented. This work points out that the time inconsistency can lead to suboptimal solutions and/or does not consider the risk in intermediate stages, proposing a methodology to calculate the suboptimality gap between a policy that was planned in a given stage for the following stages and another one that is being implemented on each date. In this last case, the algorithm is solved at each stage. From a numerical experiment the authors conclude that this gap can be ignored in extremely conservative or aggressive risk strategies, but grows as a balance between risk and return on investment is achieved.

2.3 Multistage Stochastic Programs

A generic linear T -stage stochastic programming problem has the form [35, p. 64]

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} \left\{ c_1^\top x_1 + \mathbb{E} \left[\min_{\substack{E_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} \left\{ c_2^\top x_2 + \mathbb{E} \left[\cdots + \mathbb{E} \left[\min_{\substack{E_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T \right] \right] \right\} \right] \right\} \quad (2.13)$$

where (c_1, A_1, b_1) are deterministic and a few or all entries of (c_t, A_t, E_t, b_t) , $\forall t = 2, \dots, T$, are random. The subscript in $\min\{\cdot\}$ indicates that each minimization problem is subject to the corresponding constraints, where all the entries of x_t are non-negative variables. The above problem can be solved by its deterministic equivalent form where all possible scenarios are represented by constraints. This method leads to a large problem that quickly becomes intractable in real applications, both by the exponential growth of computational time and by the exhaustion of computer memory.

An efficient way to solve multistage stochastic programs is the Stochastic Dual Dynamic Programming (SDDP) [44]. This algorithm is based on the Benders decomposition technique and approximates the future cost function using a piece linear function. For this, a set of cuts is added to the problem at each iteration. To use SDDP the premise that the scenarios are stage-wise independent should be assumed, i.e. a given realization $\omega_t \in \Omega_t$ does not depend on the history $\omega_{[t-1]}$. Then the cuts can be shared within each stage. In addition, SDDP uses a sample of paths to represent the entire scenario tree avoiding the curse of dimensionality that is common in dynamic programming techniques.

The notation used here was presented by [38]. Consider a problem (2.13) decomposed into T stages with n variables x_t and m constraints. Assume that the random right-hand-side vector $b_t(\omega_t)$ has a finite number of realizations. As it is known, the decisions made in t will influence the cost of the stage $t + 1$. This cost is defined by $Q_{t+1}(x_t, \omega_{t+1})$ and it is assumed that $Q_{T+1}(x_T, \omega_{T+1}) = 0$. The first-stage problem is

$$\begin{aligned} z = \min \quad & c_1^\top x_1 + \mathbb{E}[Q_2(x_1, \omega_2)] \\ \text{s. t.} \quad & A_1 x_1 = b_1 \\ & x_1 \geq 0 \end{aligned} \quad (2.14)$$

and for the next stages the problem is

$$\begin{aligned}
Q_t(x_{t-1}, \omega_t) = \min \quad & c_t^\top x_t + \mathbb{E}[Q_{t+1}(x_t, \omega_{t+1})] \\
\text{s. t.} \quad & A_t x_t = b_t(\omega_t) - E_t x_{t-1} \quad : \quad \lambda_t(\omega_t) \\
& x_t \geq 0
\end{aligned} \tag{2.15}$$

where $c_t \in \mathbb{R}^n$, $x_t \in \mathbb{R}^n$, A_t and E_t are $m \times n$ matrices, $b_t(\omega_t) \in \mathbb{R}^m$ and $\lambda_t(\omega_t)$ is the vector of dual variables associated to the constraints $A_t x_t = b_t(\omega_t) - E_t x_{t-1}$. Assume that the problem (2.15) has a feasible solution at stage t for all values of x_{t-1} that are feasible at stage $t - 1$.

Each iteration of the SDDP is composed by two steps called forward and backward. At the first one a total of J scenarios are sampled using Monte Carlo simulation and, starting from the root node, the recursive problem is solved for all scenarios J . By reaching the last stage, there is no uncertainty in the objective function and the problem is easily solved. A given convergence criterion is computed and in case it is not satisfied a backward step is performed. This step adds J cuts to approximate the cost-to-go function of the stage $T - 1$ and recalculate the solutions that were found for this stage in the last forward step. Then, the algorithm moves to $T - 1$ and recalculate the solution for $T - 2$, and so on until the stage $t = 0$. The term $\mathbb{E}[Q_{t+1}(x_t, \omega_{t+1})]$ is replaced by the variable θ_{t+1} and constraints

$$\theta_{t+1} - \bar{g}_{t+1,j}^\top x_t \geq \bar{h}_{t+1,j}^\top, \quad l = 1, \dots, L, \quad j = 1, \dots, J \tag{2.16}$$

are added. Thus, the problem becomes

$$\begin{aligned}
z = \min \quad & c_1^\top x_1 + \theta_2 \\
\text{s. t.} \quad & A_1 x_1 = b_1 \\
& \theta_2 - \bar{g}_{2,j}^\top x_1 \geq \bar{h}_{2,j}^\top, \quad l = 1, \dots, L, \quad j = 1, \dots, J \\
& x_1 \geq 0
\end{aligned} \tag{2.17}$$

for $t = 1$ and

$$\begin{aligned}
\tilde{Q}_t(x_{t-1}, \omega_t) = \min \quad & c_t^\top x_t + \theta_{t+1} \\
\text{s. t.} \quad & A_t x_t = b_t(\omega_t) - E_t x_{t-1} \\
& \theta_{t+1} - \bar{g}_{t+1,j}^\top x_t \geq \bar{h}_{t+1,j}^\top, \quad l = 1, \dots, L, \quad j = 1, \dots, J \\
& x_t \geq 0
\end{aligned} \tag{2.18}$$

for $t = 2, \dots, T$.

The idea behind the backward step is that once the input x_t that optimizes the term $\mathbb{E}[Q_{t+1}(x_t, \omega_{t+1})]$ in the stage $t + 1$ is known, the algorithm goes back to the stage t and adds the required cuts to get the output x_t we want. To find the set of cuts, the problem (2.18) at $t + 1$ is solved for all $\omega_{t+1} \in \Omega_{t+1}$. Let be $\bar{\lambda}_{t+1,j}^{l\top} = \mathbb{E}[\lambda_{t+1}(\omega_{t+1})]$, the gradient of the l -th cut for j is

$$\bar{g}_{t+1,j}^l = -\bar{\lambda}_{t+1,j}^{l\top} E_{t+1} \quad (2.19)$$

and the intercept is defined as

$$\bar{h}_{t+1,j}^l = \mathbb{E}[Q_{t+1}(\bar{x}_t^l(j), \omega_{t+1})] + \bar{\lambda}_{t+1,j}^{l\top} E_{t+1} \bar{x}_t^l(j) \quad (2.20)$$

where $\bar{x}_t^l(j)$ is the solution computed in the last forward pass.

The algorithm stops when the lower bound of z (called \underline{z}) is sufficiently close to the average of the costs among the scenarios, that is called upper bound or \bar{z} . It can be calculated by

$$\bar{z} = \frac{1}{J} \sum_{j=1}^J \sum_{t=1}^T c_t^\top \bar{x}_t^l(j) \quad (2.21)$$

Thus, the convergence test with $100(1 - \beta)\%$ confidence level proposed by [44] is

$$\underline{z} > \bar{z} - z_{\beta/2} \frac{\sigma}{\sqrt{J}} \quad (2.22)$$

where $z_{\beta/2}$ comes from standard normal distribution and σ is the standard deviation of the costs, i.e.

$$\sigma = \sqrt{\frac{1}{J} \sum_{j=1}^J \left(\sum_{t=1}^T c_t^\top \bar{x}_t^l(j) \right)^2 - \bar{z}^2}. \quad (2.23)$$

A pseudocode for SDDP is shown in Algorithm 1. Remark that the problem (2.18) always has a feasible solution at t for all feasible values of x_{t-1} at $t - 1$. Artificial variables with penalty coefficients can be added in the objective function to ensure this premise. More details on the implementation of SDDP can be seen in [38], where the authors also propose that the algorithm be executed up to a fixed number of iterations.

Algorithm 1 SDDP algorithm

- 1: **for** $t \leftarrow 2$ to T **do**
- 2: $\theta_t \leftarrow -\infty$

```

3: end for
4:  $L \leftarrow 0$ 
5:  $l \leftarrow 1$ 
6: procedure FORWARD
7:    $t \leftarrow 1$ 
8:   Solve problem (2.17)
9:   for  $j \leftarrow 1$  to  $J$  do
10:      $\bar{x}_1^l(j) \leftarrow x_1$ 
11:   end for
12:   for  $t \leftarrow 2$  to  $T$  do
13:     for  $j \leftarrow 1$  to  $J$  do
14:        $x_{t-1} \leftarrow \bar{x}_{t-1}^l(j)$ 
15:       Solve problem (2.18)
16:     end for
17:   end for
18: end procedure
19: procedure CONVERGENCETEST
20:    $\underline{z} = z$ 
21:   Compute  $\bar{z}$  from equation (2.21)
22:   Compute  $\sigma$  from equation (2.23)
23:   if  $\underline{z} > \bar{z} - \frac{z_{\beta/2}}{\sqrt{J}}\sigma$  then
24:     Stop
25:   else
26:     Call Backward
27:   end if
28: end procedure
29: procedure BACKWARD
30:   for  $t \leftarrow T$  to 2 do
31:     for  $j \leftarrow 1$  to  $J$  do
32:       for  $\omega_t \in \Omega_t$  do
33:         Solve problem (2.14)
34:       end for

```

```

35:           Compute the  $l$ -th cut from equations (2.15) and (2.16)
36:       end for
37: end for
38:    $L \leftarrow L + 1$ 
39:    $l \leftarrow l + 1$ 
40:   Call Forward
41: end procedure

```

In [45] a few modifications in the standard SDDP are discussed, like other methods for sampling the set of scenarios and a stopping criterion based on statistical hypothesis testing. In [6] it is proposed a methodology to obtain the upper limit of the risk estimation problem using AVaR, which can be used in more efficient stopping rules for algorithms such as SDDP, as demonstrated. In [46] a combination between SDDP and the L-shaped method is presented in order to obtain temporal dependence and ensure consistency. Nested decomposition is a technique very similar to SDDP, except that it does not use scenario sampling. The forward pass is solved by considering the whole scenario tree and, therefore, the problem may become intractable more easily. In general, convergence is achieved when the difference between \bar{z} and \underline{z} is lower than a given tolerance. An example of its use can be found in [47], where a problem that maximizes the commercialization of small hydroelectric power plants is solved.

Other works deserve attention. [4] uses an algorithm based on Benders decomposition and sampling by importance for solving a multistage asset selection problem with linear programming. The work [7] propose a general multistage model for the wealth allocation problem based on an analytical metric to limit the risk. The asset returns were obtained from different methodologies for scenario tree generation, such as GBM and GARCH(1,1) and GARCH(2,2), and the numerical results were compared with the classical VaR approach. For all cases the analytical technique proved to be satisfactory as a measure of risk, being particularly useful in situations where computational time is a limited resource.

In [48] a risk-averse stochastic hydrothermal planning problem is solved considering constraints that limit the risk associated with the energy deficit. The author uses Lagrangian relaxation for the decomposition of the problem and adds AVaR

constraints in the standard SDDP. The problem is solved for three stages and each node of the scenario tree was branched into five children, resulting in 125 scenarios in the last stage. The formulation consists of limiting the energy deficits with a previously defined risk level, aiming to minimize costs and to achieve reliability in the energy supply.

The work [49] formulates a multistage mixed integer programming model for the operation of hydrothermal systems, in which compares criteria to make convex the cost-to-go functions using a non-traditional focus of the Lagrangian relaxation technique of recursive constraints. Three methodologies are used to compare the results: the first one considering the Lagrange multipliers obtained by linear relaxation of the original problem; the second considering the multipliers obtained from the solution of a local convex problem; and the third considering the second approach with a search procedure for updating the multipliers. Such problems, even when they are small or medium size, suffer from the curse of dimensionality. For this reason, decomposition techniques are more adequate to solve this kind of problem, since many smaller problems are solved instead of a single and complex problem.

Chapter 3

Wealth Allocation Problem

An agent has n assets in which he can invest an initial amount of money W_0 over a period of time $[0, T]$. The asset prices are modeled as a stochastic process, denoted by $S_t \in \mathbb{R}^n$, for $t \in \mathcal{D} := \{0, 1, 2, \dots, T\}$, where the i -th entry S_t^i represents the prices of the i -th asset at time t . This process determines a filtration $\mathcal{F}_{\mathcal{D}} = \{\mathcal{F}_t\}_{t \in \mathcal{D}}$. Each element in \mathcal{D} represents a trading day. Among all these trading days there exists several dates $\mathcal{T} := \{0 = t_0 < t_1 < t_2 < \dots < t_K < T\}$ at which one is allowed to perform some investment operations like selling or buying assets shares, according to some conditions.

The objective is to determine an investment strategy, represented by a $\mathcal{F}_{\mathcal{T}}$ -adapted stochastic process $x_t \in \mathbb{R}^n$, where the entry x_t^i represents the number of shares of i the agent has at time t . From these two processes, the amount of money W_t owned by the investor at time t can be determined as follow

$$W_t = S_t^\top x_{[t]_{\mathcal{T}}}, \quad \text{where} \quad [t]_{\mathcal{T}} := \max\{t' \in \mathcal{T} : t' \leq t\}. \quad (3.1)$$

The process x_t must satisfy operational conditions that will be mentioned as basic constraints.

3.1 Basic Constraints

Assume the investment strategy x satisfies the basic constraints if $x \in \mathcal{B}$. Here the set \mathcal{B} is described by a number of constraints intended to model basic aspects which the investment policy should satisfy. A few examples are discussed below.

Positions: To restrict the investment strategy only to long positions, constraints of the form $x_t \geq 0, \forall t \in \mathcal{T}$ should be included. Of course, it is possible to devise more sophisticated policies where it is possible to allow short positions at a few dates.

Initial Portfolio: The meaning of this constraint is clear: to ensure that only W_0 is invested initially, i.e.

$$W_0 = S_0^\top x_0. \quad (3.2)$$

Self-financing Portfolio: This constraint aims to ensure that along all the investment period the investor will not be able to add or withdraw money, so

$$S_{t_k}^\top x_{t_{k-1}} = S_{t_k}^\top x_{t_k}, \quad \forall k = 1, 2, \dots, K. \quad (3.3)$$

These constraints lead to the following standard structure which will serve as the basis for other more complete models where the risk and return of the policies are managed.

$$\begin{aligned} \min \quad & f(x) \\ \text{s.a.} \quad & S_0^\top x_0 = W_0 \\ & S_{t_k}^\top (x_{t_k} - x_{t_{k-1}}) = 0, \quad \forall k = 1, 2, \dots, K \\ & x_t \geq 0, \quad \forall t \in \mathcal{T} \end{aligned} \quad (3.4)$$

For solving the multistage stochastic problem it is possible to use some especial techniques, otherwise, because of size problem, it easily becomes prohibited. In order to use special methods like SDDP, the problem is required to have an special structure. For ensuring a proper structure it is necessary to make a change of variables whereby: $w_t^i(\omega_t) = S_t^i(\omega_t)x_t^i(\omega_t)$. Since $x_t^i(\omega_t)$ is the number of shares owned by agent of the asset i , $w_t^i(\omega_t)$ represents the amount of money he has invested on that asset. Then

$$W_t(\omega_t) = \sum_i w_t^i(\omega_t). \quad (3.5)$$

An important fact about the process x_t is that the model allows changes only at dates $t \in \mathcal{T}$ and so if $t_1, t_2 \in \mathcal{T}, t_1 < t_2$, then $x_{t_1} = x_t$ for $t_1 \leq t < t_2$. This important fact can be modeled by adding other constraints. Denoting by R_t^i the return of S^i

at time t (i.e. $R_t^i := (S_t^i - S_{t-1}^i)/S_{t-1}^i$), it is easy to verify that $S_t^i = (1 + R_t^i)S_{t-1}^i$. So $x_{t_1} = x_t$ for $t_1 \leq t < t_2$ holds if and only if $w_t^i = (1 + R_t^i)w_{t-1}^i$, for $t = t_1 + 1, \dots, t_2 - 1$. Using this new variables $w_t^i \geq 0$ it is possible to rewrite all conditions of \mathcal{B} in the model. For example, the initial portfolio constraint just means $W_0 = \sum_i w_0^i$. The self-financing constraint can be expressed as

$$\sum_i w_{t_k-1}^i (1 + R_{t_k}^i) = \sum_i w_{t_k}^i, \quad \text{for } k = 1, 2, \dots, K. \quad (3.6)$$

A convenient way of expressing all the constraints of the model is by defining the sets

$$\mathcal{X}_0 = \left\{ w_0 : W_0 = \sum_i w_0^i \right\} \quad (3.7)$$

and

$$\mathcal{X}_t(w_{t-1}, R_t) = \begin{cases} \{w_t : \sum_i w_{t-1}^i (1 + R_t^i) = \sum_i w_t^i\}, & \text{if } t \in \mathcal{T}; \\ \{w_t : w_t^i = (1 + R_t^i)w_{t-1}^i \forall i\}, & \text{otherwise.} \end{cases} \quad (3.8)$$

With this notation, the multistage problem can be expressed as

$$\begin{aligned} \min \quad & f(w_0, \dots, w_T) \\ \text{s.t.} \quad & w_0 \in \mathcal{X}_0 \\ & w_t \in \mathcal{X}_t(w_{t-1}, R_t) \quad \text{for } t = 1, 2, \dots, T \end{aligned} \quad (3.9)$$

and, obviously, it is possible to include more constraint to each set \mathcal{X}_t whenever necessary to ensure gains or to limit losses.

3.2 Model 1

A first approach is to maximize the expected value of the portfolio at the end of the investment period T . This can be made by replacing the objective function of (3.9) by $\mathbb{E}[W_T]$ with negative sign, since the problem is of minimization. Note that the implementation of the modeling handled so far would guide the strategies in which all money is applied to the highest return action between the dates t_{k-1} and t_k , considering the process S_t . In choosing the most profitable assets the agent is also exposed to the great risks that accompany them, since the correlation between risk and return in the real problem tends to be positive. This may lead to large

losses and inviability of the investment. Thus, it is necessary to implement risk handling policies using constraints.

The model will observe the returns of the portfolio along the investment period and will search for policies that, in some sense, can avoid large losses. For two consecutive trading dates $t - 1$ and t , the portfolio return can be defined by the expression

$$R_t = \frac{W_t - W_{t-1}}{W_{t-1}}. \quad (3.10)$$

From this definition note that R_t is \mathcal{F}_t -measurable. At a trading date t , since the return R_{t+1} is uncertain, a risk measure can be used for estimating the risk of having losses at that day. Intuitively, this means anticipating the information revealed in $t + 1$ for the decision to be made in t . Using a conditional risk measure it is possible to set a limit to the potential loss.

A good choice for this model is to consider the risk measures separately at each stage. Although the function (2.10) is not time consistent for all cases when it is used to minimize the risk (as seen in the Section 2.2), it has some advantages that can be exploited, especially in this case when the policies maximize the return by limiting the risk rather than minimizing losses. First, it is possible to determine a sequence of real numbers $\phi_0, \phi_1, \dots, \phi_{T-1} \in [0, 1]$ representing the thresholds to be adopted at each stage. This corresponds to the following constraints

$$\rho_t(-R_{t+1}) \leq \phi_t, \text{ w.p.1, } \forall t = 1, \dots, T - 1. \quad (3.11)$$

Secondly, this measure maintains the proper structure to use the standard SDDP directly in the case where $\rho_t = \text{CAVaR}_\alpha$. If the filtration \mathcal{F}_t is finite, it is easy to verify that for $Z_{t+1} \in \mathcal{M}_{t+1}$ and $Z_t \in \mathcal{M}_t$ with $Z_t \geq 0$, then

$$\text{CAVaR}_\alpha(Z_t Z_{t+1} | \mathcal{F}_t) = Z_t \text{CAVaR}_\alpha(Z_{t+1} | \mathcal{F}_t). \quad (3.12)$$

Since $R_{t+1} = \frac{W_{t+1} - W_t}{W_t}$, with W_t being \mathcal{F}_t measurable, it is possible to write

$$\text{CAVaR}_\alpha(-R_{t+1} | \mathcal{F}_t) = \frac{W_t + \text{CAVaR}_\alpha(-W_{t+1} | \mathcal{F}_t)}{W_t} \quad (3.13)$$

thus, the constraint

$$\text{CAVaR}_\alpha(-R_{t+1} | \mathcal{F}_t) \leq \phi_t \quad (3.14)$$

is equivalent to

$$CAVaR_\alpha(-W_{t+1}|\mathcal{F}_t) \leq (\phi_t - 1)W_t \quad (3.15)$$

for $t = 1, \dots, T - 1$. By the equation (2.9), the above constraint can be expressed as a linear programming problem for each $\omega_t \in \Omega_t$ (see the formulation of [30]). This approach leads to a linear model that allows the standard SDDP solves large-scale instances. Model 1 is formulated as follows, where the risk constraints appear highlighted but obviously can be added to \mathcal{X}_t . Remark that SDDP assumes stage-wise independence.

$$\begin{aligned} \min \quad & -\mathbb{E}[W_T] \\ \text{s.t.} \quad & w_0 \in \mathcal{X}_0 \\ & w_t \in \mathcal{X}_t(w_{t-1}, R_t), \forall t = 1, \dots, T \\ & \min_{u_{\omega_t}} \{u_{\omega_t} + \frac{1}{\alpha} \mathbb{E}[(-W_{t+1} - u_{\omega_t})^+]\} \leq (\phi_t - 1)W_t, \forall t = 1, \dots, T - 1 \end{aligned} \quad (3.16)$$

3.3 Model 2

Another approach is to minimize the risk of the investment. Thus, it is possible to take advantage of SDDP's cost-to-go functions to simplify more complex risk measures, such as nested and composite measures. In the first case a strategy is to minimize the average risk over time, i.e. $\tilde{\rho}(-R_1, -R_2, \dots, -R_T)/T$. By the definition of $\tilde{\rho}$ presented in (2.11), this function is equivalent to

$$\begin{aligned} \frac{1}{T} \tilde{\rho}(-R_1, -R_2, \dots, -R_T) &= \frac{1}{T} \rho_1 \circ \rho_2 \circ \dots \circ \rho_T(-\Sigma_t R_t) \\ &= \frac{1}{T} \rho_1 \left(-R_1 + \rho_2(-R_2 + \dots + \rho_T(-R_T)) \right). \end{aligned} \quad (3.17)$$

Thus, for the case $\rho_t = CAVaR_\alpha$, the previous function corresponds to

$$\frac{1}{T} CAVaR_\alpha \left(-R_1 + CAVaR_\alpha(-R_2|\mathcal{F}_1 + \dots + CAVaR_\alpha(-R_T|\mathcal{F}_{T-1})) \right). \quad (3.18)$$

However, as demonstrated in [38], it is not possible to use SDDP directly to minimize nested risk measures and this can have a negative impact on algorithm efficiency. Thus, it is interesting to use a composite measure which does not have this limitation and is also time consistent. Unlike the previous model, the linearization of the risk measure is not as simple it was in equation (3.15). To overcome this problem it is possible to use the absolute profit

$$P_t = W_t - W_{t-1} \quad (3.19)$$

instead of return R_t . Although this alternative does not bring advances from the theoretical point of view, it is valid to see the behavior of the strategy and it also allows the computational advantages of using linear programming.

The objective function using a composite risk measure $\hat{\rho}(-P_1, \dots, -P_T)$ with $\rho_t = CAVaR_\alpha$ is

$$\begin{aligned} \min \left\{ \min_{u_0} \left\{ u_0 + \alpha^{-1} \mathbb{E}[(-P_1 - u_0)^+] + \mathbb{E} \left[\min_{u_1} \left\{ u_1 + \alpha^{-1} \mathbb{E}[(-P_2 - u_1)^+ | \mathcal{F}_1] + \dots \right. \right. \right. \right. \\ \left. \left. \left. + \mathbb{E} \left[\min_{u_{T-1}} \left\{ u_{T-1} + \alpha^{-1} \mathbb{E}[(-P_T - u_{T-1})^+ | \mathcal{F}_{T-1}] \right\} | \mathcal{F}_{T-2} \right] \dots \right\} | \mathcal{F}_1 \right] \right\}. \end{aligned} \quad (3.20)$$

In order to provide a better control over the model it is necessary to include a few constraints to ensure that the policies lead to a minimum wealth percentage set by the investor. Again, a sequence of real numbers $\varphi_1, \dots, \varphi_T \in [0, 1]$ can be given to represent this wealth throughout the stages. Obviously, values larger than one are also possible, but this results in very strong impositions on the model and can lead to infeasibility, since this condition has to be guaranteed in any scenario, including the most pessimistic ones. Thus, for each $\omega_t \in \Omega_t$, these kind of constraints should be included

$$W_t \geq \varphi_t W_{t-1}. \quad (3.21)$$

Note that $(1 - \varphi_t)$ can be understood as a limit to the maximum loss to which the agent is exposed in percentage terms. This is an important metric and makes it possible to compare both models, since their objective functions are different. For example, after obtaining investment policies by resolving models with given values for ϕ_t in Model 1 and $(1 - \varphi_t)$ in Model 2, it is possible to use historical data or simulate new scenarios to see if these limits are being respected.

Model 2 is expressed as

$$\begin{aligned} \min \left\{ \min_{u_0} \left\{ u_0 + \alpha^{-1} \mathbb{E}[(-P_1 - u_0)^+] + \mathbb{E} \left[\min_{u_1} \left\{ u_1 + \alpha^{-1} \mathbb{E}[(-P_2 - u_1)^+ | \mathcal{F}_1] + \dots \right. \right. \right. \right. \\ \left. \left. \left. + \mathbb{E} \left[\min_{u_{T-1}} \left\{ u_{T-1} + \alpha^{-1} \mathbb{E}[(-P_T - u_{T-1})^+ | \mathcal{F}_{T-1}] \right\} | \mathcal{F}_{T-2} \right] \dots \right\} | \mathcal{F}_1 \right] \right\} \end{aligned}$$

$$\begin{aligned}
\text{s.t. } & w_0 \in \mathcal{X}_0 \\
& w_t \in \mathcal{X}_t(w_{t-1}, R_t), \forall t = 1, \dots, T \\
& W_t \geq \varphi_t W_{t-1}, \forall t = 1, \dots, T
\end{aligned} \tag{3.22}$$

and, of course, the constraint (3.21) can be included to each set \mathcal{X}_t .

Chapter 4

Methodology

The methodology used in the case study is composed by three steps: I. Assets selection; II. Scenario tree generation; and III. Portfolio optimization. In the first one, the fifty shares that compose the IBrX-50 index for the four-month period between May and August 2016 were considered. These were arranged vertically in the Table 4.1 following descending order of tradings. For example, the ABEV3 share was the one with the highest number of tradings in the period, followed by PETR4, and so on until the SMLE3 share.

Table 4.1: Assets considered in this work.

| | | | | |
|-------|-------|--------|-------|-------|
| ABEV3 | VALE3 | BBSE3 | VIVT4 | BRAP4 |
| PETR4 | BBAS3 | LREN3 | SUZB5 | QUAL3 |
| ITSA4 | CIEL3 | BRML3 | HYPE3 | EGIE3 |
| ITUB4 | CCRO3 | CSNA3 | SBSP3 | RADL3 |
| PETR3 | GGBR4 | GOAU4 | CPFE3 | EQTL3 |
| BBDC4 | TIMP3 | WEGE3 | MRVE3 | NATU3 |
| VALE5 | BRFS3 | UGPA3 | BRKM5 | PCAR4 |
| BVMF3 | CMIG4 | LAME4 | ESTC3 | CSAN3 |
| JBSS3 | EMBR3 | USIM5 | CTIP3 | MULT3 |
| KROT3 | BBDC3 | KLBN11 | FIBR3 | SMLE3 |

The returns of these assets were simulated using the GBM. This choice was based on two factors. Firstly because it does not create dependency between the stages and this is an important premise for using SDDP. For this same reason the variable

used was the returns R_t instead of the stock prices S_t . As shown in the Section 2.1, knowing the parameters μ' and Σ' and generating samples N_k , the returns are calculated only by setting a time interval Δt . This does not occur with the prices S_t , which are a function of the values obtained in the previous stage. In addition, to ensure independence property, this approach also keeps the scenario tree more compact. Consider, for example, that each stage has three possible scenarios of returns. The tree representing these returns grows at the ratio $3(T - 1)$ and the stock prices tree grows exponentially at rate 3^{T-1} , since each node represents a possible combination between R_t e R_{t-1} . Note that the term $T - 1$ appears because the tree starts from a single root node in both cases, i.e. when $t = 0$. The Figure 4.1 allows us to view this simple example.

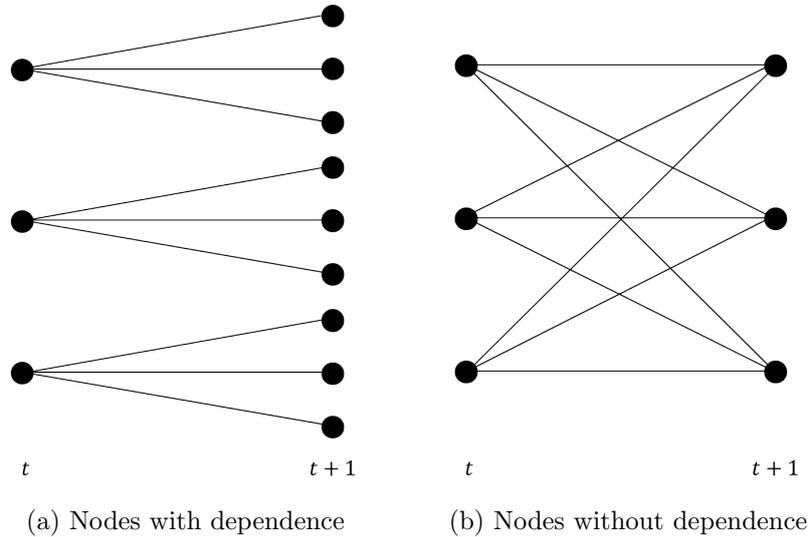


Figure 4.1: Structure of the scenario tree with and without dependence.

The second reason for using GBM is that the investment strategy of this study is short/medium term, while more sophisticated models for prices volatility are needed in long term situations. The period of two years between 2/1/2014 and 1/31/2016 was adopted for the collection of historical data and evaluation of μ' and Σ' . Stages were defined monthly, starting in February 2016 ($t = 0$) and ending in January 2017 ($t = T = 11$), and $\Delta t = 21$ days approximately, as it is common in the financial market. For each stage were generated $|\Omega_t| = 30$ simulations of R_t with the same probability. The Figure 4.2 shows the average rate of these returns for all fifty assets considered in this study over time, only as an exemplification proposal. It should be

interpreted as a single scenario among all those that are possible for the problem.

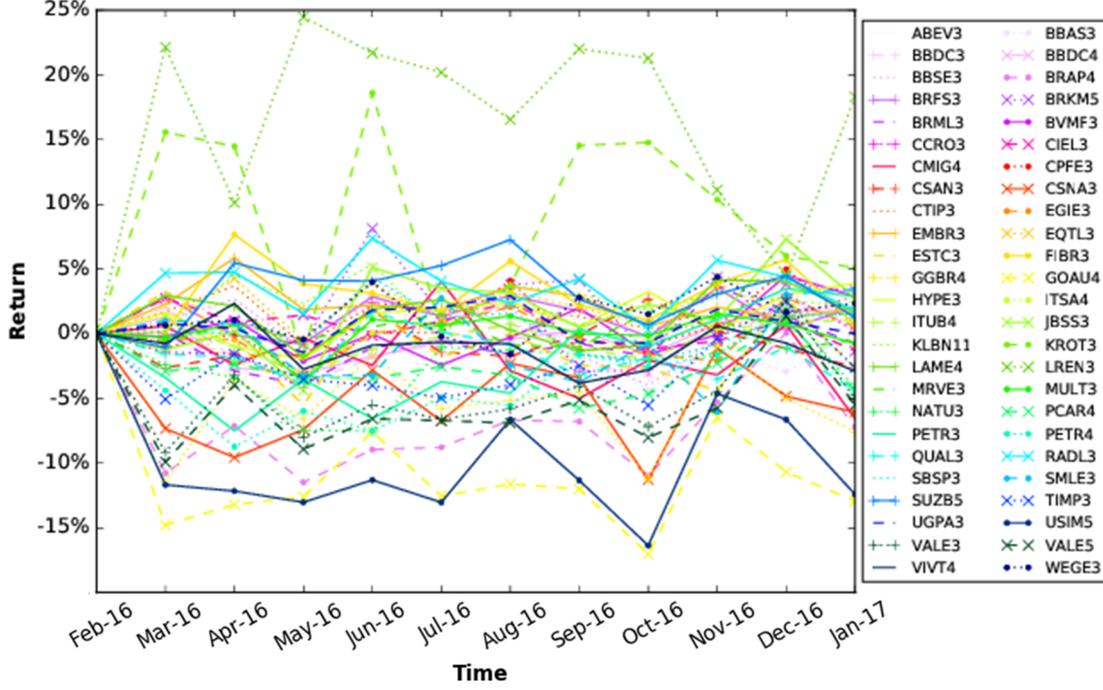


Figure 4.2: Assets average returns.

To approximate the prices of these assets using SG, it is necessary to consider the sigma-algebras are finite. The entries R_t^i and w_t^i are \mathcal{F}_t -measurable and they can be represented as vectors $(R_{t,a}^i)_{a \in \Omega_t}$ and $(w_{t,a}^i)_{a \in \Omega_t} \in \mathbb{R}^{|\Omega_t|}$, respectively. Thus, let $R_{t,a} = (R_{t,a}^1, R_{t,a}^2, \dots, R_{t,a}^n)$ and $w_{t,a} = (w_{t,a}^1, w_{t,a}^2, \dots, w_{t,a}^n) \in \mathbb{R}^n$, the self-financing constraint (when $t \in \mathcal{T}$) becomes

$$\sum_{i=1}^n w_{t_k-1,a}^i (1 + R_{t_k,b}^i) = \sum_{i=1}^n w_{t_k,b}^i, \quad \forall k = 1, \dots, M, a \in \Omega_{t_k-1}, b \in \Omega_{t_k} \quad (4.1)$$

and if $t \notin \mathcal{T}$ it is only needed to add constraints that update the amount applied in each share, i.e.

$$w_{t_k,b} = (1 + R_{t_k,b}^i) w_{t_k-1,a}^i, \quad a \in \Omega_{t_k-1}, b \in \Omega_{t_k}. \quad (4.2)$$

Also, note that for a vector $Z_{t+1} = (Z_{t+1,a})_{a \in \Omega_{t+1}}$ it is possible to write, for each element $a \in \Omega_t$

$$CAVaR_\alpha(Z_{t+1} | \mathcal{F}_t)_a = \min_{u_a} \left\{ u_a + \frac{1}{\alpha} \sum_{b \in \mathcal{C}_a} p_{b|a} [Z_{t+1,b} - u_a]^+ \right\} \quad (4.3)$$

where $p_{b|a}$ is the conditional probability of b given the node a . Assuming R_t is stage-wise independent, then the term $p_{b|a}$ is replaced only by the occurrence probability

of b , i.e. p_b . Thus, the constraint (3.15) is equivalent to

$$u_{t,a} + \frac{1}{\alpha} \sum_{b \in \Omega_{t+1}} p_b [-W_{t+1,b} - u_{t,a}]^+ \leq (\phi - 1)W_{t,a}, \quad a \in \Omega_t \quad (4.4)$$

where extra variables $u_{t,a}$ were added. This expression can be linearized by the following group of constraints

$$\begin{cases} u_{t,a} + \alpha^{-1} \sum_{b \in \Omega_{t+1}} p_b v_{t+1,b} \leq (\phi - 1)W_{t,a}, & a \in \Omega_t; \\ v_{t+1,b} \geq -W_{t+1,b} - u_{t,a}; \\ v_{t+1,b} \geq 0. \end{cases} \quad (4.5)$$

Finally, it is easy to see that constraints (3.21) can be written as $W_{t+1,b} \geq \varphi_t W_{t,a}$, where $a \in \Omega_t$ and $b \in \Omega_{t+1}$. With these modifications, all constraints are adequate for the implementation of SDDP. The Bellman equations for the objective function of Model 1 are as follows:

$$Q_T(w_{T-1}, R_T) = \min \left\{ - \sum_{b \in \Omega_T} p_b W_{T,b} \right\} = \min \left\{ - \sum_{b \in \Omega_T} p_b \left(\sum_{i=1}^n (1 + R_{T,b}^i) w_{T-1,a}^i \right) \right\} \quad (4.6)$$

for $t = T$, and

$$Q_t(w_{t-1}, R_t) = \min \left\{ - \mathbb{E}[Q_{t+1}(w_t, R_{t+1})] \right\} \quad (4.7)$$

for $t = 1, \dots, T - 1$.

In Model 2, they are written as:

$$\begin{aligned} Q_T(w_{T-1}, R_T, u_{T-1}) &= \min_{w_{T,b}} \alpha^{-1} \sum_{b \in \Omega_t} p_b v_{T,b} \\ \text{s.t.} \quad v_{T,b} &\geq -P_{T,b} - u_{T-1,a} \\ v_{T,b} &\geq 0 \end{aligned} \quad (4.8)$$

for the last stage, and

$$\begin{aligned} Q_t(w_{t-1}, R_t, u_{t-1}) &= \min_{w_{t,b}, u_{t,b}} \alpha^{-1} \sum_{b \in \Omega_t} p_b v_{T,b} + u_{t,b} + \mathbb{E}[Q_{t+1}(w_t, R_{t+1}, u_t)] \\ \text{s.t.} \quad v_{T,b} &\geq -P_{T,b} - u_{T-1,a} \\ v_{T,b} &\geq 0 \end{aligned} \quad (4.9)$$

for $t = 1, \dots, T - 1$.

SDDP was implemented in Matlab using the toolbox FAST [50]. FAST can solve linear multistage stochastic programs using solvers like Cplex and Gurobi. Basically, the program should be divided into two scripts, one containing the general configuration parameters of SDDP and another for the formulation of the optimization problem containing the recursive functions. At each iteration, $J = 1000$ paths were sampled with Monte Carlo and the convergence test adopted was of 95% of confidence ($\beta = 0.05$). Similarly, the confidence level for CAVaR in the models was also 95% ($\alpha = 0.05$) and a initial capital $W_0 = \text{R\$ } 100.00$ was adopted, but all performance analysis in this study were made in relative terms so they are not conditioned on the amount invested initially.

Operations like selling or buying assets shares were allowed every two months, then $\mathcal{T} := \{0, 2, 4, 6, 8, 10\}$. This point deserves attention. FAST, as well as other SDDP implementations, provides the solution only for the root node of the tree. This is a common feature of multistage stochastic programs, since the main interest is to make a decision at the present moment. Then, moving forward in time and getting to the point where a new decision is needed, it is possible to make choices based on new data that has occurred over this period. Real-world decisions are also made this way. But when solving the problem it is often possible to make simulations about the future and through them get solutions for the next stages. In FAST, these simulations must be paths of the scenario tree (such as the set J).

At this point, two operation strategies can be proposed. In the first one, the models are solved using SDDP and sample paths to obtain the policies in all trading dates $t \in \mathcal{T}$. Here 5000 trials were performed. Note that the problem was solved only once and a planning was made for the entire investment period – obviously the portfolio can change at each trading date. This is especially useful in a context where there is computational limitation. The other operation strategy is to run the SDDP every time a decision is needed, i.e. when $t \in \mathcal{T}$. Thus, the number of stages gets smaller because the start point forward in time and it is necessary to generate a new tree with the information that has become known between t_{k-1} and t_k . For this, the stock prices from 2/1/2016 to 1/31/2017 were collected and the problem was solved again at each trading date. The closest path to actual returns was used to obtain the solution between t_{k-1} and $t_k - 1$ and to update the capital value W_{t_k} .

available for investment, i.e.

$$W_{t_k} = \sum_{i=1}^n w_{t_{k-1}}^i (1 + R_{t_k}^{*i}), \quad \text{for } k = 1, 2, \dots, K \quad (4.10)$$

where $R_{t_k}^{*i}$ is the return of the i -th asset on the real data closest path between $k - 1$ and k . The comparison of the paths with the real returns was made taking the Euclidean distance between them. Assume the first operation strategy is defined as “planned” and the second one as “adjusted”, because in this last case newest information is used to run the problem again.

Finally, different thresholds were adopted for the losses. Model 1, described in (3.16), was solved with $\phi = \{0.05, 0.10, 0.15\}$. The thresholds for Model 2 were $\varphi = \{0.90, 0.95, 1.00\}$ – see formulation (3.22). This resulted in twelve cases, six for the planned strategy and six for the adjusted strategy. The performance of the policies was evaluated using real data, in order to verify what would have been the real investment return if the portfolio compositions found in this study were adopted. All tests occurred on a laptop computer with a 2.4GHz quad-core processor and 8GB of RAM. The implementation was done in Matlab R2013b using FAST 0.9.1b and Gurobi 7.0 for optimization of linear problems. A code in Python 2.7 was developed for the analysis of the numerical results.

Chapter 5

Numerical Results

The results were analyzed from three points of view: The composition of the portfolio; if the maximum losses remained under the adopted thresholds; and the cumulative returns of the obtained policies. As the two proposed models seek different solutions – the objective functions, for example, measure different variables – these analysis make the comparison between them more appropriate.

5.1 Portfolio Analysis

Portfolio charts show the percentage of capital invested per asset on each trading date. For accurate values, see the Appendix A. The solutions between the planned and adjusted strategies were placed side by side in order to identify the proximity between portfolios. In the planned case, the portfolio composition for future dates is the average among the compositions found in the 5000 simulations. Two aspects related to investment diversification deserve attention.

The first is that the number of shares in the portfolio was higher in the intermediate stages and lower in the initial and final stages. A few explanations for this behaviour can be given. It is reasonable that the initial portfolio is less diversified because the uncertainty at this instant is lower than in the next stages; for the last decision, note that December is a less busy period in many areas of the economy and this contributes to a somewhat more stable environment, reducing risk and allowing investment to be concentrated to increase returns. The second aspect is that the level of diversification remained the same in both models. This may be a result of

the proximity between risk-return settings. In other words, if Model 1 maximizes the value of the portfolio by limiting the risk of losses and Model 2 minimizes the risk of the investment by requiring a minimum return, these approaches may be closely depending on the values chosen for ϕ and φ .

In Model 1, the policies between planned and adjusted strategies were significantly different from each other (with the exception of the initial portfolio, of course). On a few occasions the assets chosen have been repeated and even in these cases their importance in the portfolio has changed (see decisions made in April, for example). This is associated with how distant the scenarios were to reality, but it does not necessarily mean a bad feature. As was said, the main purpose of SG is not to predict the future but to provide good decisions. Note that the portfolio compositions also varied greatly depending on the different values for ϕ , indicating that the solutions are very sensitive to the choice of this threshold. As an exception, the portfolio from the adjusted strategy had a greater closeness degree in the decisions made in October and December. On this last date in particular, the selected assets was approximately the same for all values of ϕ . As expected, the diversification when $\phi = 0.15$ (see Figure 5.3) is slightly lower than in other cases, since risk tolerance was higher.

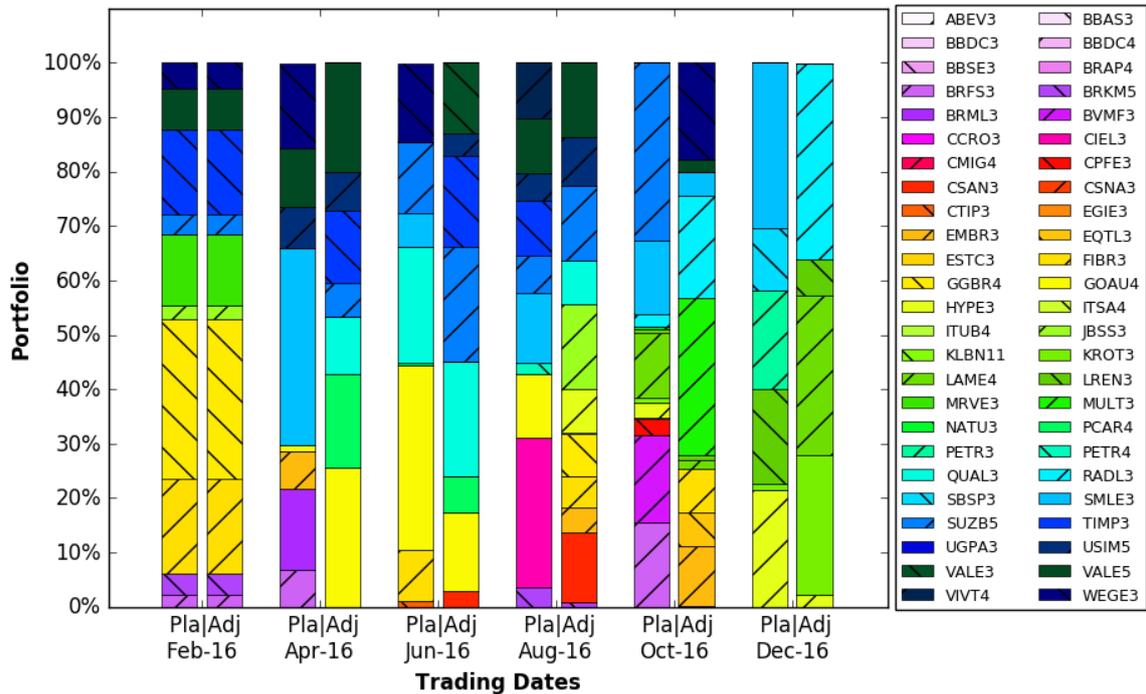


Figure 5.1: Portfolio composition of Model 1 with $\phi = 0.05$.

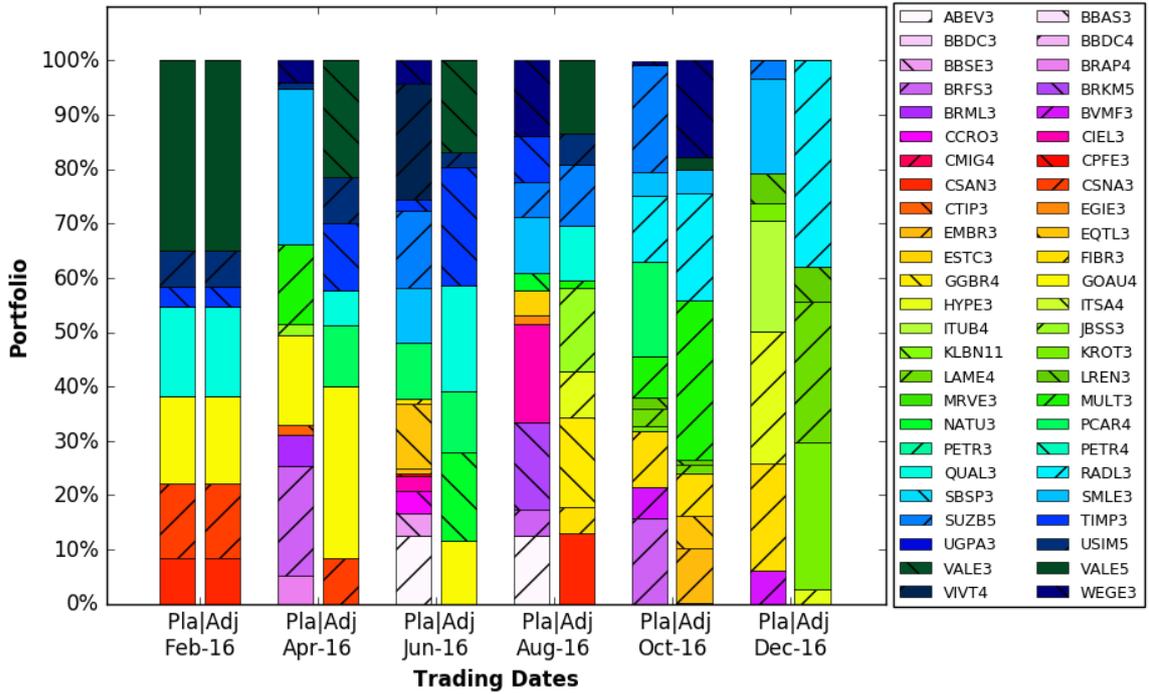


Figure 5.2: Portfolio composition of Model 1 with $\phi = 0.10$.

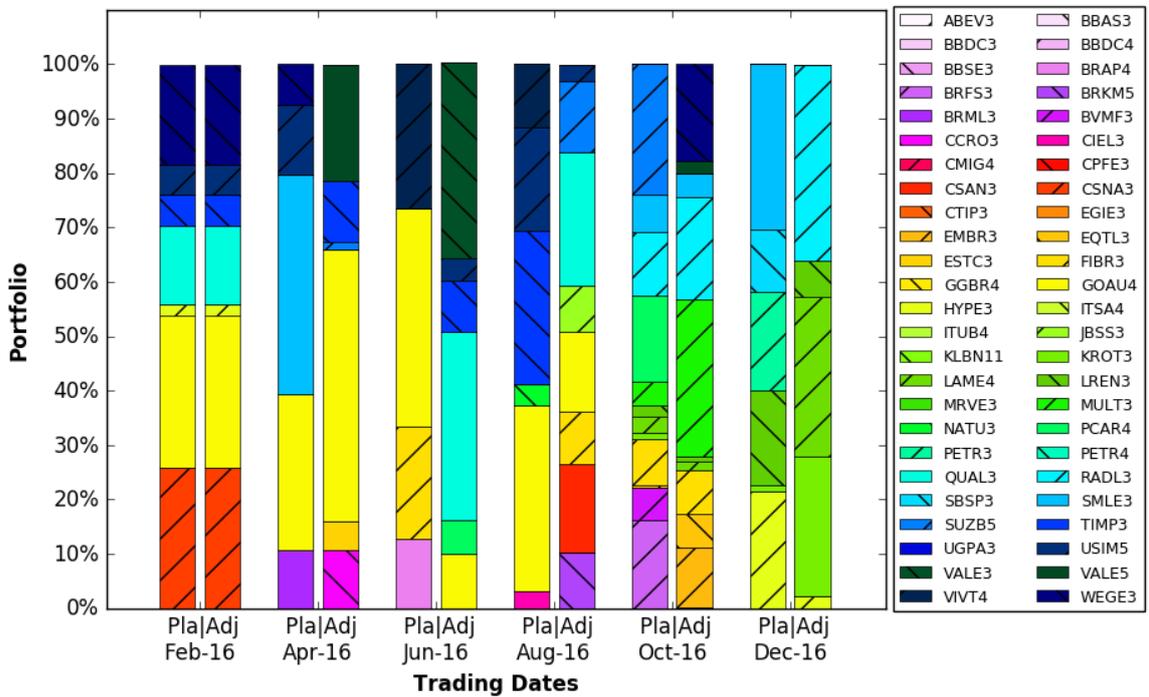


Figure 5.3: Portfolio composition of Model 1 with $\phi = 0.15$.

Analogously to the previous model, the investment decisions of Model 2 presented a great variation between both the planned and adjusted strategies, and as a

function of the values for φ . Observe that a time consistent risk measure is used in this case and, according to one of the definitions presented earlier, planned policies for the future should be implemented. However, note that the results do not consider the decisions of both strategies in each node of the tree, because in the planned case the average of the solutions was adopted in each trading date. In addition, the constraints (3.21) are very strong and must be guaranteed for all scenarios, even those that will not occur. Differently, time consistency is addressed in the literature in models that only optimize the risk [36] or the convex combination between it and the portfolio value [43] – therefore, they are not influenced by constraints that determine the return on investment. Finally, the existence of multiple optimal solutions was not investigated.

It is worth to highlight the similarity between the adjusted portfolios in December when $\varphi = 0.90$ and $\varphi = 0.95$ (Figures 5.4 and 5.5, respectively), which allows us to infer that the least risky assets that will result in a maximum loss of 10% of the portfolio in this two-month period are necessarily the same. Again, the decisions for the last date obtained with adjusted strategy are very close to each other and also with Model 1.

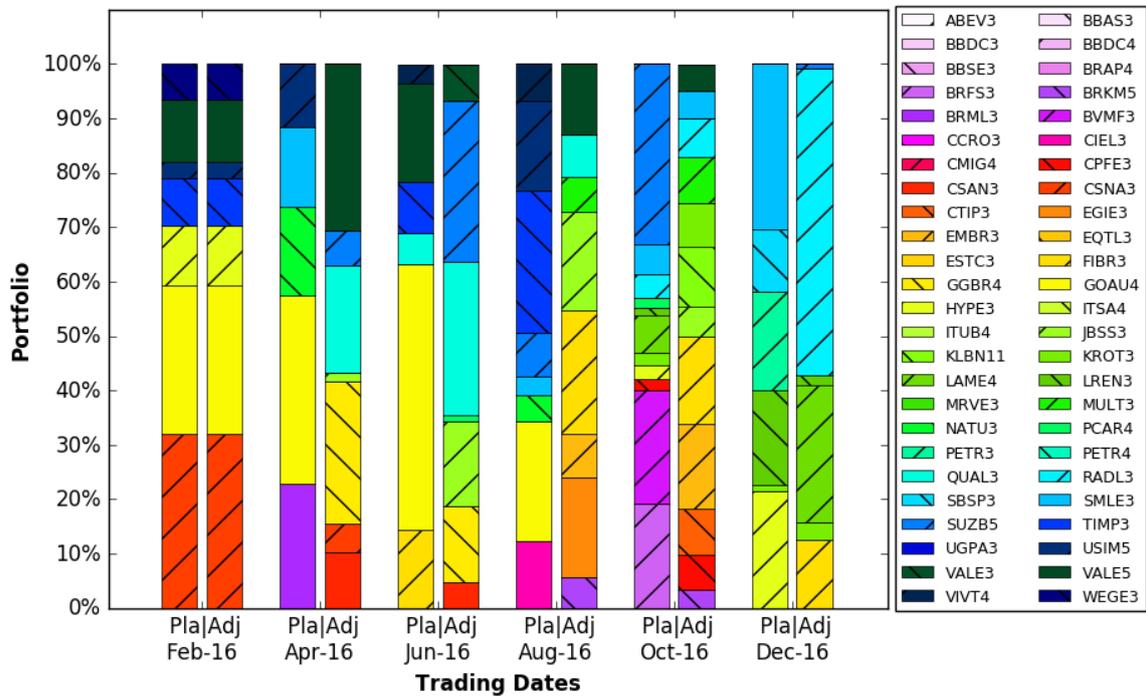


Figure 5.4: Portfolio composition of Model 2 with $\varphi = 0.90$.

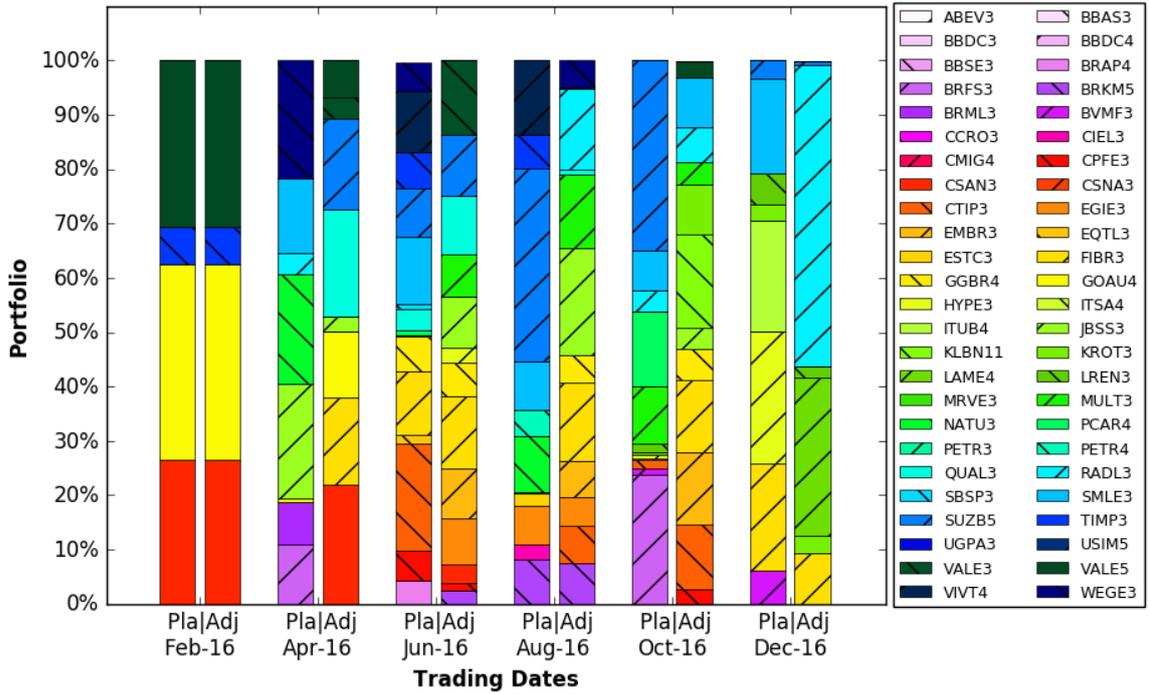


Figure 5.5: Portfolio composition of Model 2 with $\varphi = 0.95$.

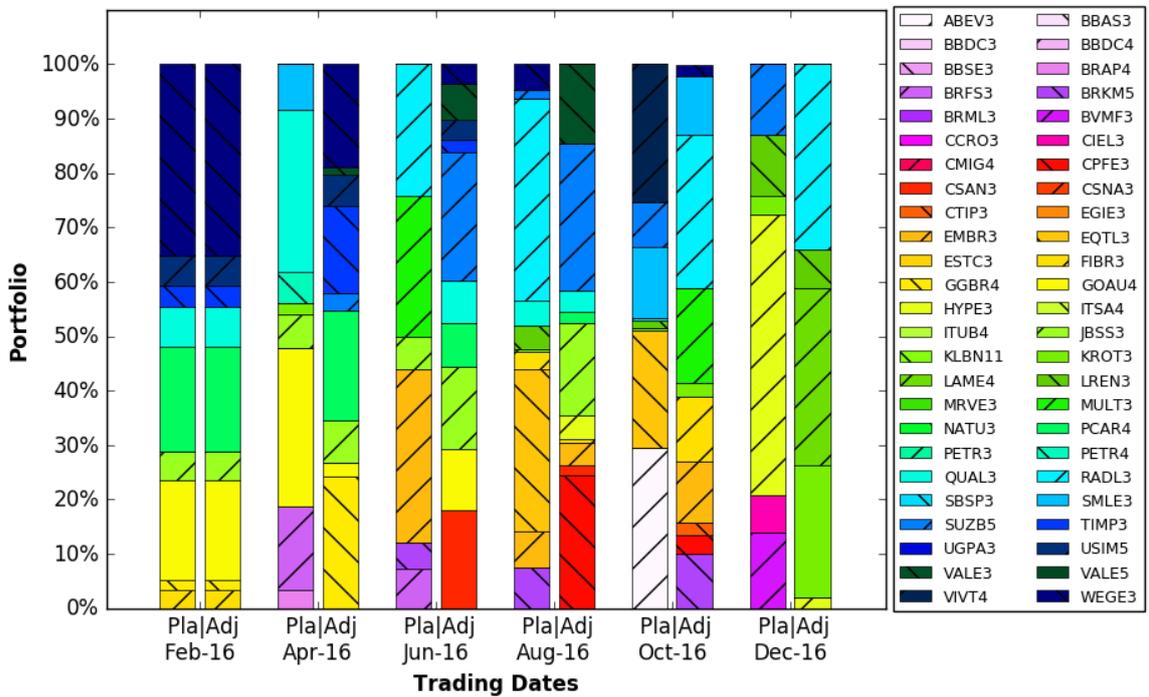


Figure 5.6: Portfolio composition of Model 2 with $\varphi = 1.00$.

5.2 Losses Analysis

The figures below show the capital losses. The variable used was

$$-R_t = -\frac{W_{t+1} - W_t}{W_t} \quad (5.1)$$

and note that these values correspond to the losses calculated by the real returns of the assets. An interesting result is that when the thresholds were exceeded this occurred in April and once in November (see Figure 5.12). The losses that occurred in other months are not relevant. As this result is shared by both models and both operating strategies, it is possible to infer that SG was not able to represent the extremely pessimistic scenario that actually occurred in May.

The best performance was from planned policies on Model 1. Except when $\phi = 0.05$ (Figure 5.7), where the loss reached approximately 11.1%, all thresholds were respected. This did not occur with the adjusted strategy, where the portfolio value reduced 22.3%, 29.6% and 26.2% in April for the cases $\phi = 0.05$, $\phi = 0.10$ and $\phi = 0.15$, respectively.

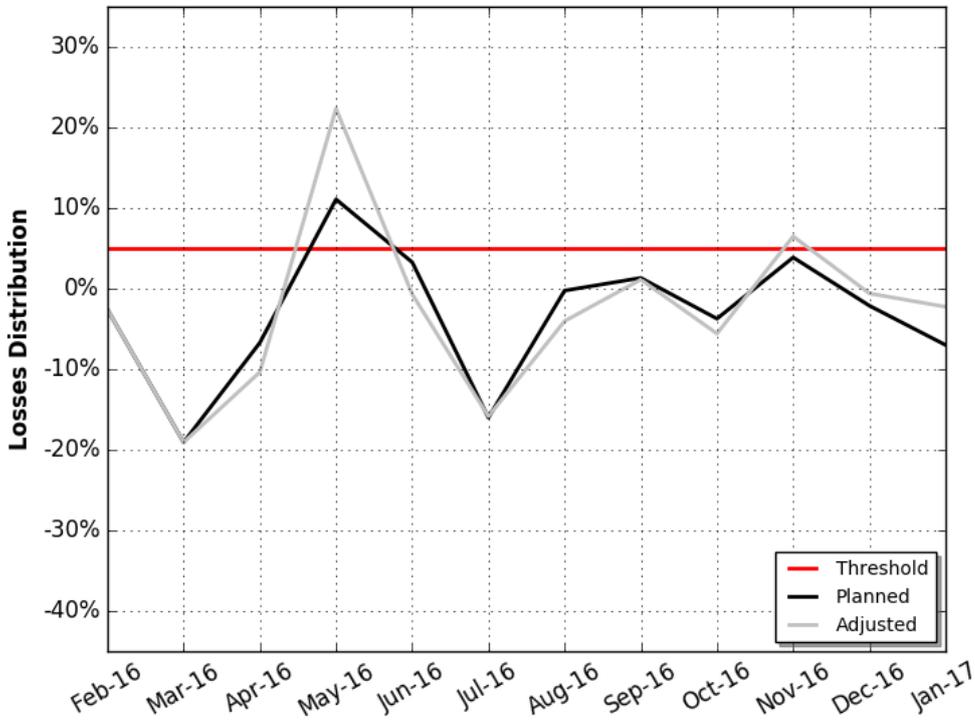


Figure 5.7: Losses of Model 1 with $\phi = 0.05$.

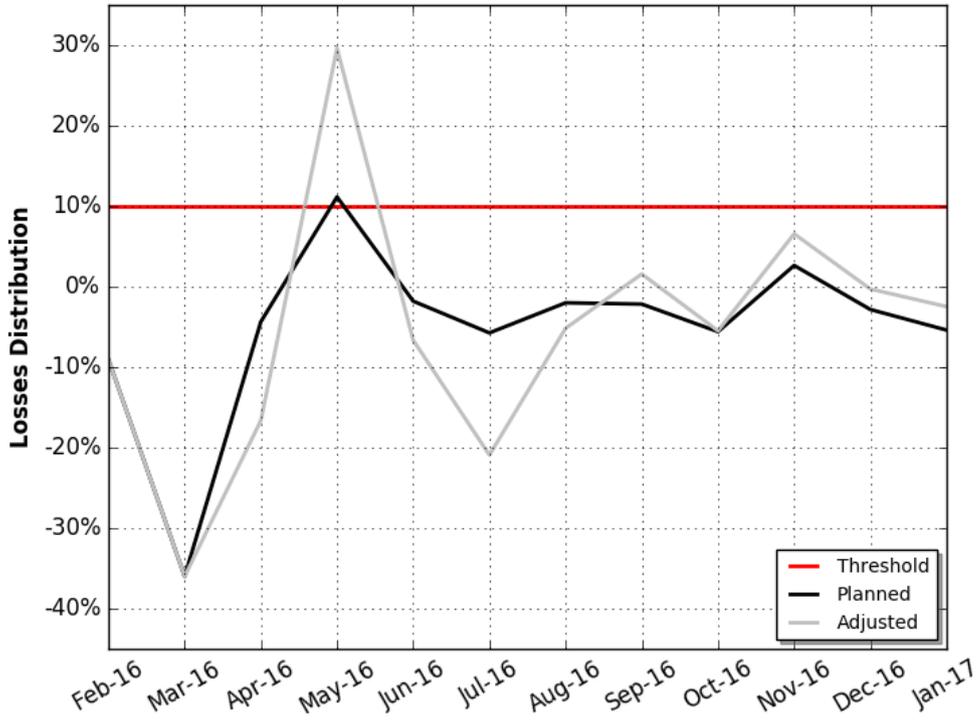


Figure 5.8: Losses of Model 1 with $\phi = 0.10$.

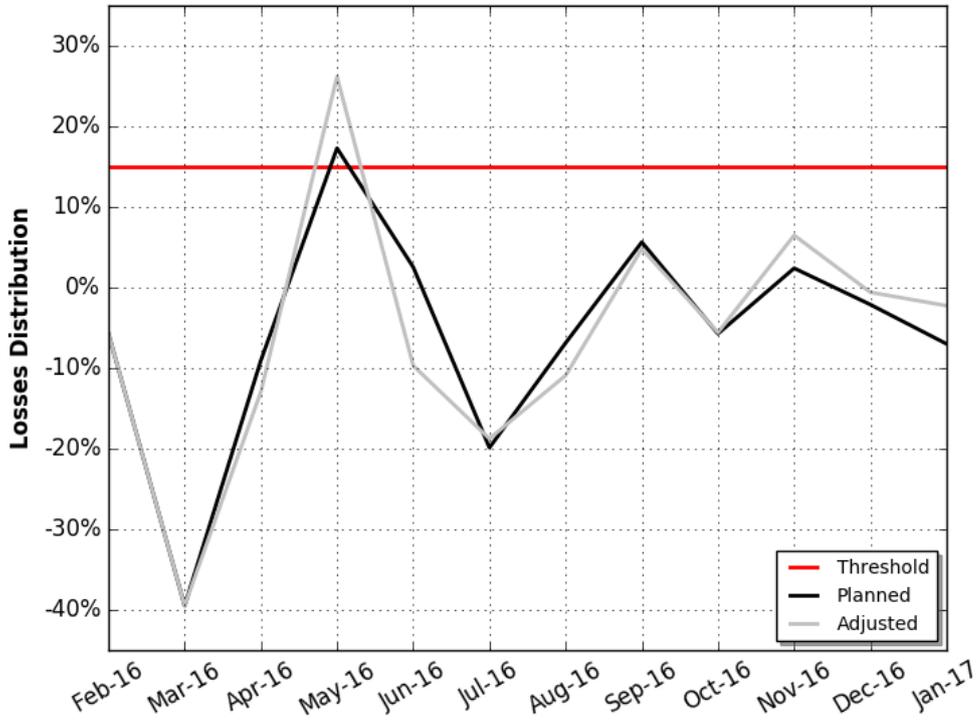


Figure 5.9: Losses of Model 1 with $\phi = 0.15$.

In Model 2 the limit $1 - \varphi$ can be used for the losses. In the second case, when $\varphi = 0.95$ (Figure 5.11), no threshold was exceeded. In the other two cases the losses occurred using both planned and adjusted strategies. For $\varphi = 0.90$ (Figure 5.10) the capital reduction of the adjusted portfolio was 20.7% in April while the loss of the planned policy was 21.6%.

Finally, for $\varphi = 1.00$, losses are not allowed in any scenario. It is clear that this is a very strong requirement on the model. Consequently, only in this case the losses occurred in other dates than April (see Figure 5.12). However, they were smaller when compared with the other cases. Note that in Model 2 the losses between planned and adjusted strategies are closer than in Model 1. Then, it is also expected that the returns show this same behavior.

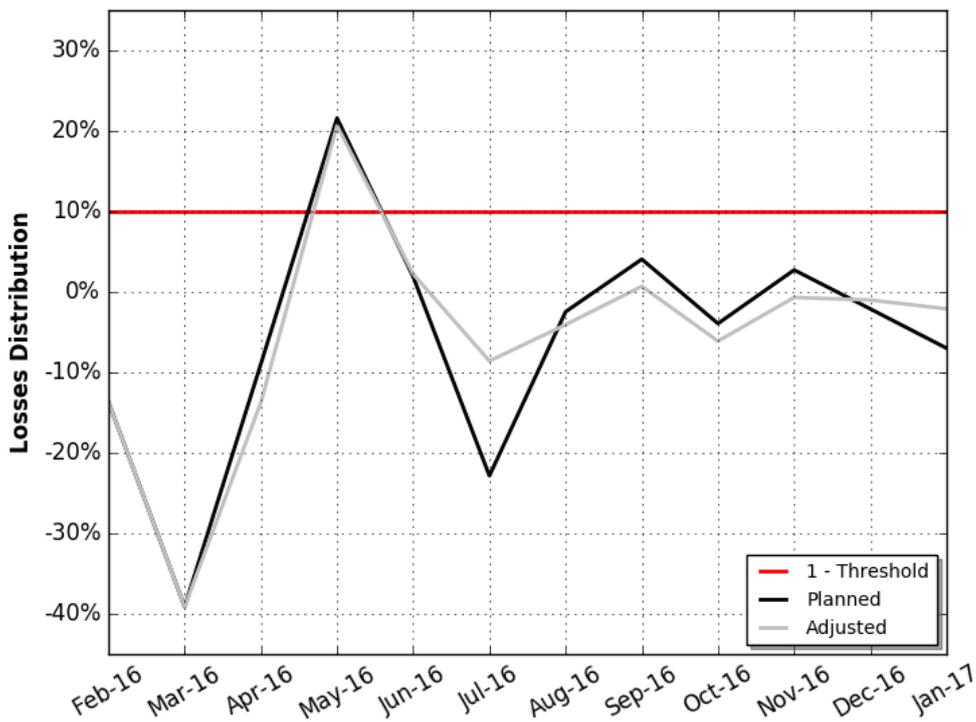


Figure 5.10: Losses of Model 2 with $\varphi = 0.90$.

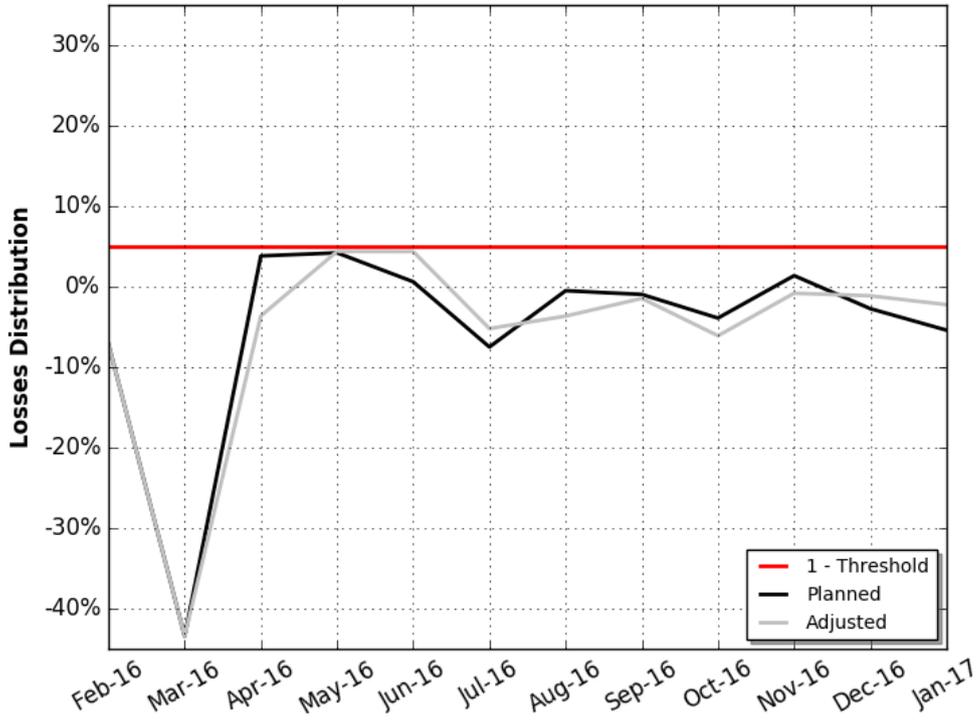


Figure 5.11: Losses of Model 2 with $\varphi = 0.95$.

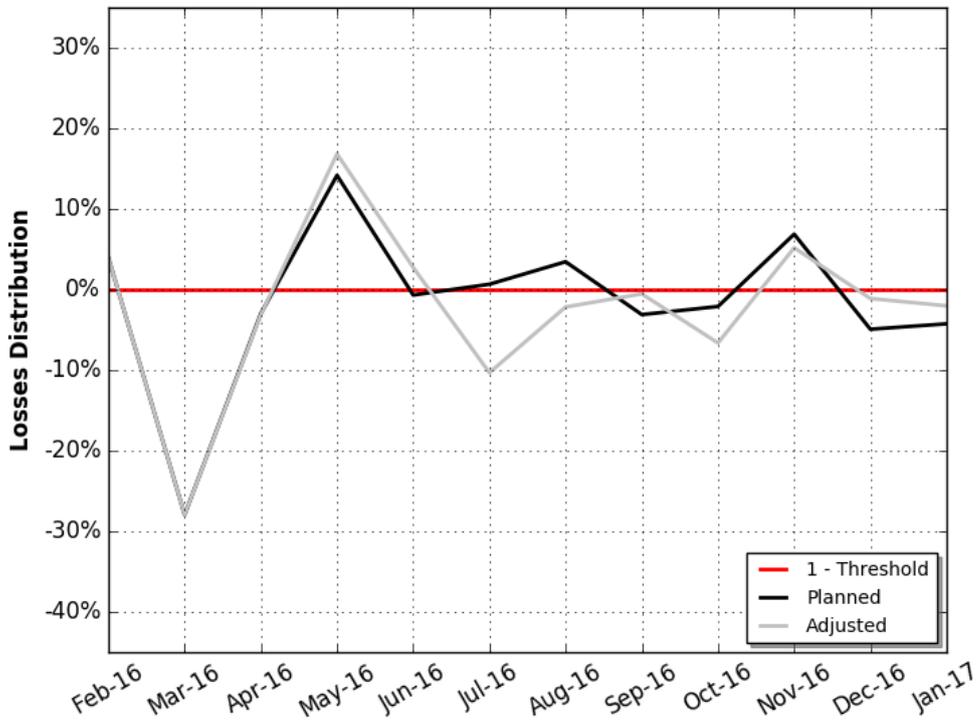


Figure 5.12: Losses of Model 2 with $\varphi = 1.00$.

5.3 Returns Analysis

This section shows the cumulative return over the investment period, which represents the real gain of the agent if the portfolios found here were actually used. The cumulative return at t is computed as

$$\frac{R_t - R_0}{R_0}. \tag{5.2}$$

Analyzing these series, it is evident that the planned strategy is the best one in Model 1. This can be justified by the end-of-investment return, which was higher than the results of adjusted strategy in all the cases (see Figures 5.13, 5.14 and 5.15).

Besides, the adjusted strategy seems to be more sensitive to downturn periods as it occurs in April. Except when $\phi = 0.15$, where the risk-tolerance is higher, the return of the planned portfolio is stable. This does not occur with the adjusted strategy. On the other hand, it seems to have a better performance after the downturn period, recovering money more quickly.

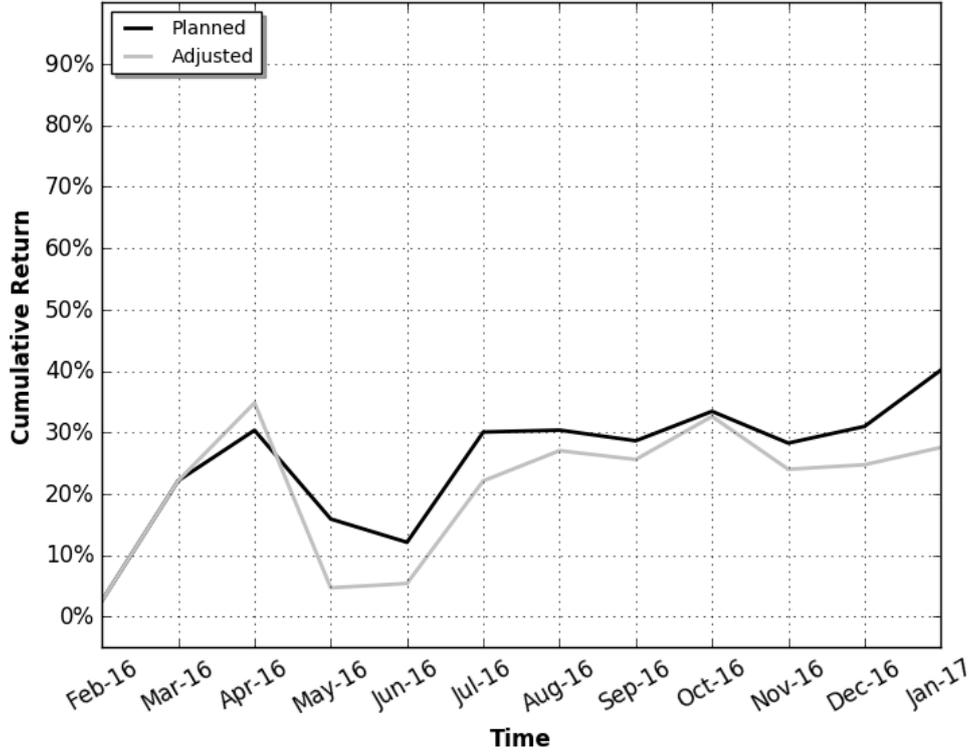


Figure 5.13: Cumulative returns of Model 1 with $\phi = 0.05$.

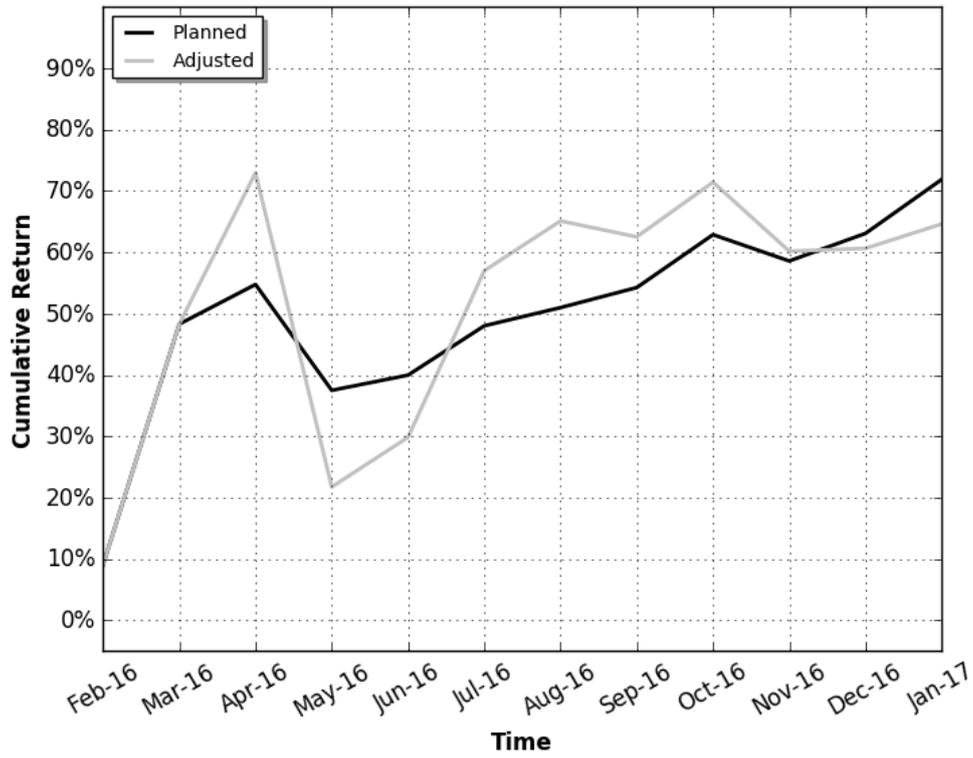


Figure 5.14: Cumulative returns of Model 1 with $\phi = 0.10$.

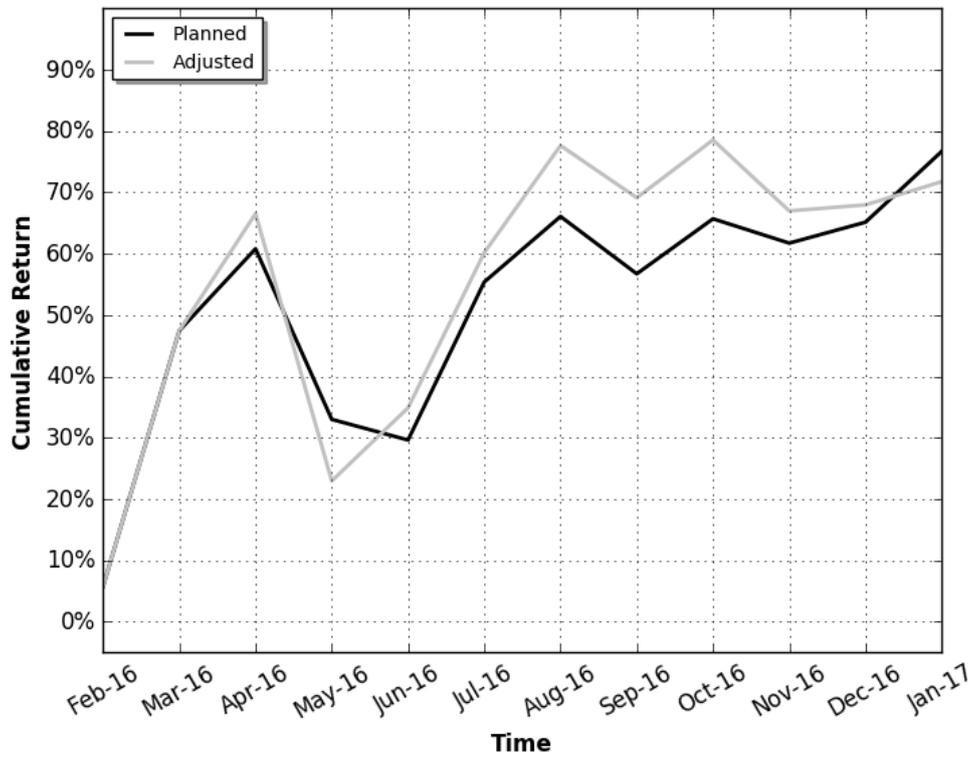


Figure 5.15: Cumulative returns of Model 1 with $\phi = 0.15$.

In Model 2 the adjusted portfolios was better along all the investment and achieved higher returns in the last period. These results show that the independent CAVaR works better if combined with the planned strategy and the multiperiod composite measure works better with the adjusted strategy. In this study the time inconsistent risk measure was more appropriated for the situation where the problem was solved at once and the portfolio follow the planning done. If the measure is time consistent it will be better when it is possible to use the newest information and solve the problem again. In each trading date, a new tree is generated to run the SDDP but only the root node solution matters.

Therefore, only the risk measure with time consistency achieved the advantages of using data from the scenario that actually occurred to adjust the problem. These results can be faced with [43], where the authors encourage to use of these measures when it is possible to solve the problem in each stage. Differently from this work, here it is not coherent to evaluate the gap between the planned and the adjusted investment because the returns are being calculated directly by real values and not by the objective functions. Besides, the planned policies were derived from simulations while the gap is computed considering the paths of the tree.

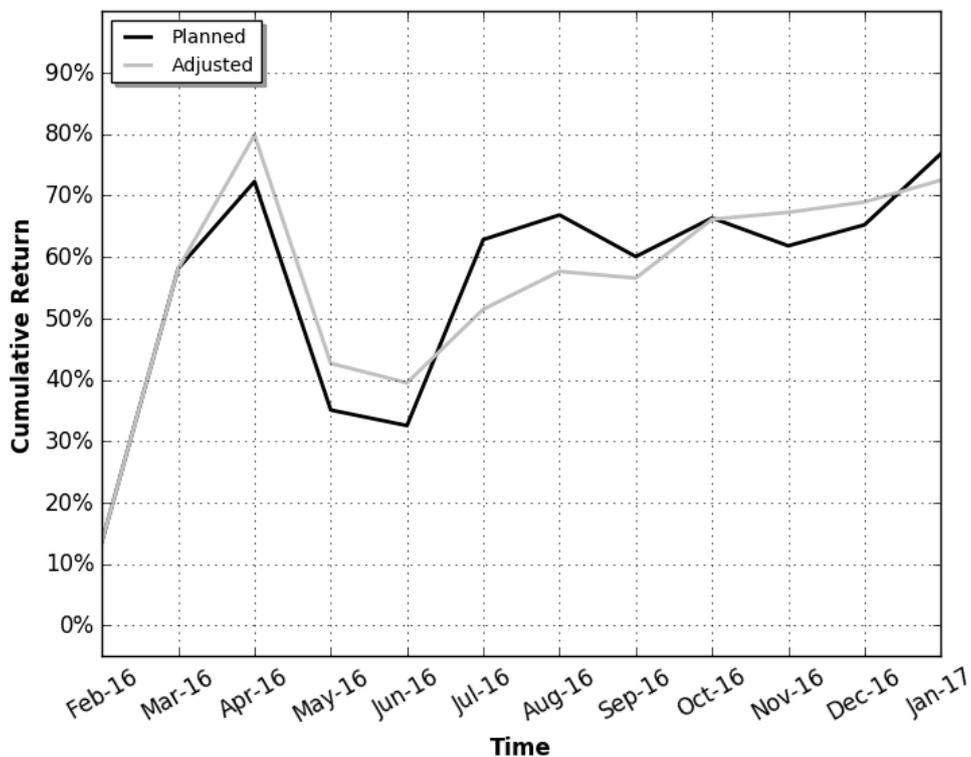


Figure 5.16: Cumulative returns of Model 2 with $\varphi = 0.90$.

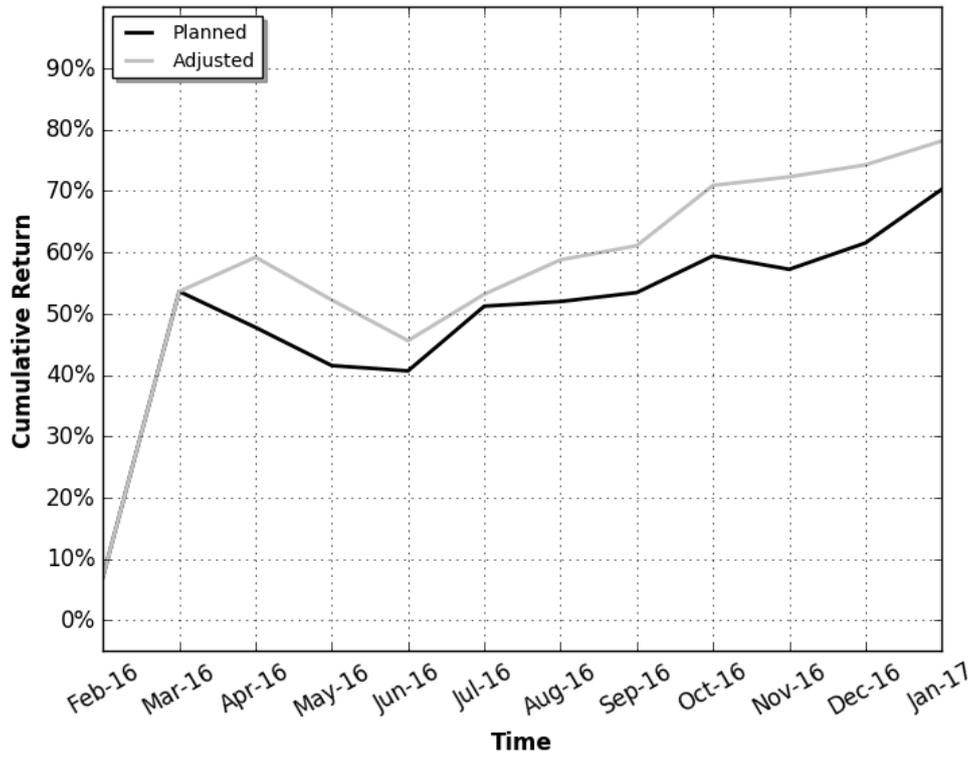


Figure 5.17: Cumulative returns of Model 2 with $\varphi = 0.95$.

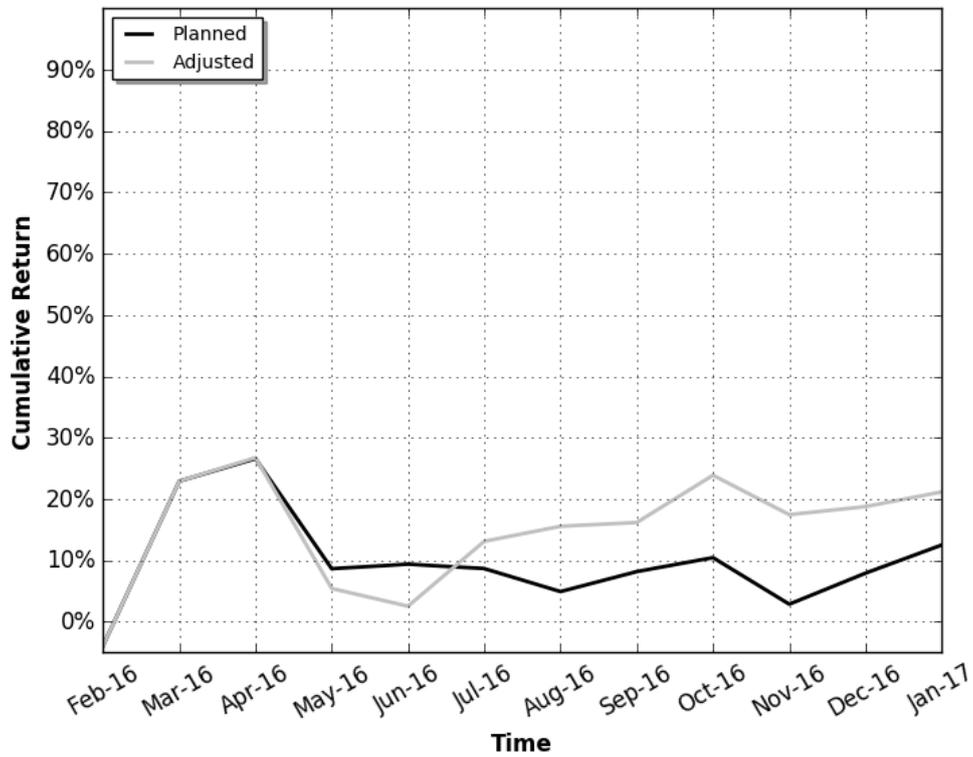


Figure 5.18: Cumulative returns of Model 2 with $\varphi = 1.00$.

As a basis for comparison, the portfolio values were also computed using the IBrX-50 index and a naive $1/n$ portfolio, where the applied capital is equally divided among all the assets. The results for the planned and adjusted strategies are summarized in the Figures 5.19 and 5.20, respectively. Note that Model 1 is able to obtain higher returns at the end of the investment more frequently, which could be explained by the objective functions of the two formulations. Observe that, in this model, the expected value of the portfolio in the last stage is maximized. In Model 2 the policies minimize the risk, leading to smaller losses along the investment period in all the cases.

In both operation strategies the four best cases are the same: Model 1 with $\phi = 0.10$ and $\phi = 0.15$ and Model 2 with $\varphi = 0.90$ and $\varphi = 0.95$, but alternate their positions. Analogously, the general ranking of both models follow this same behavior and a particular case changes only one position from one strategy to the other. But this does not mean that is convenient to ignore the operation type. Instead, it is essential to determine which case has the best performance – considering that the agent will chose one of these thresholds.

Note that, in the planned strategy, the Model 1 with $\phi = 0.10$ is the best approach because it got the third higher cumulative return and also kept the losses under the threshold. Thus, when the agent wishes to plan *a priori* decisions of buying or selling assets shares including the next stages, this could be a good choice. When it is possible solve the problem containing all the available information and only the decision to be taken at present moment matters, the Model 2 with $\varphi = 0.95$ is the better approach. It had the higher return on investment while ensure a more stable behavior along the period. Note that this case also suffered less in the worst date (April). However, as previously stated, this is a strong constraint and not always can lead to feasible solutions. In fact, the real-world decisions usually occur using the most recent data, but both operation strategies have practical application and can be justified in specific contexts.

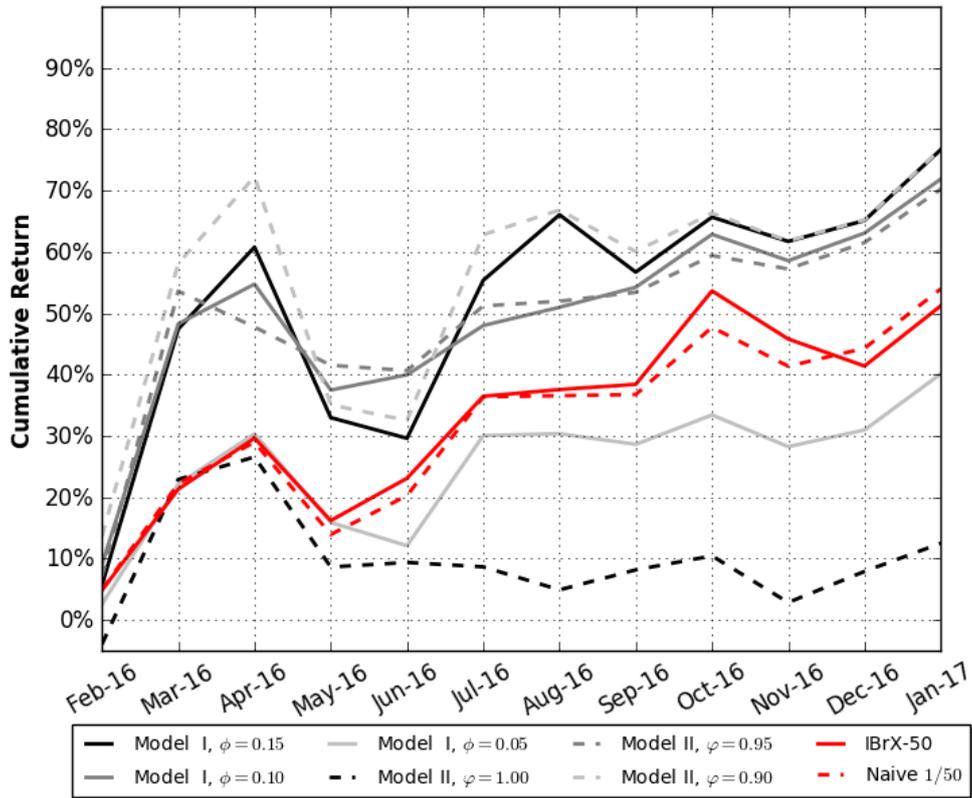


Figure 5.19: Cumulative returns from planned strategy.

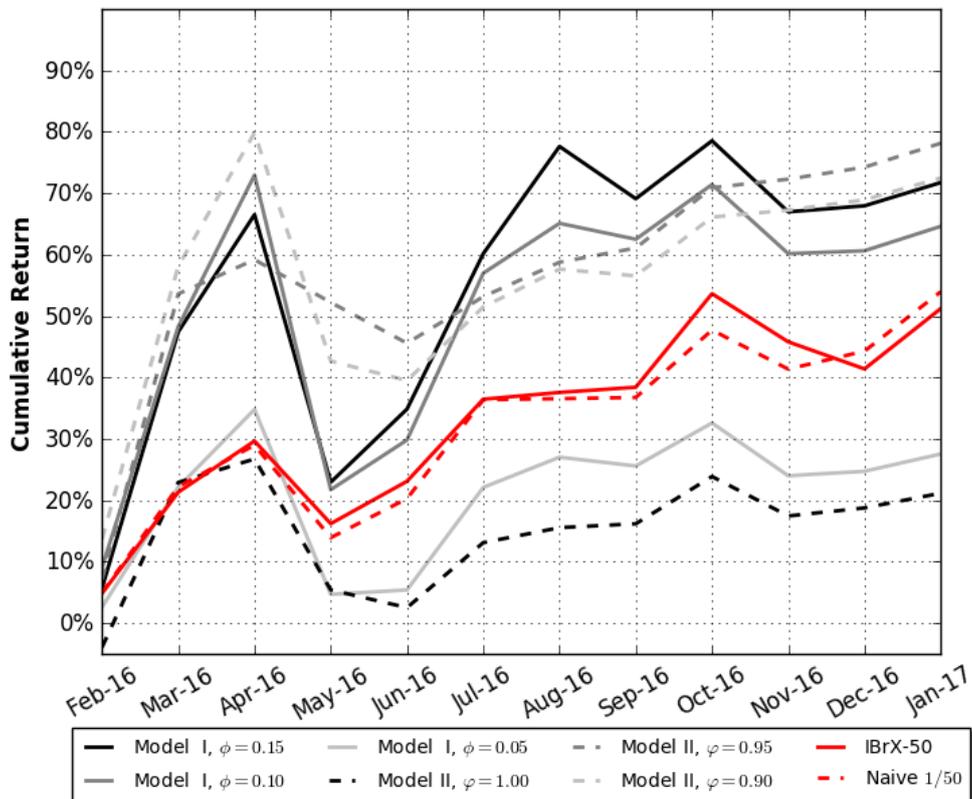


Figure 5.20: Cumulative returns from adjusted strategy.

Table 5.1 shows the return of investment for all the case studies. Observe that these values correspond to the real portfolio gain and, as can be seen, it was possible to obtain returns that are much higher than the IBrX-50 index. Also, note that it is simply the cumulative return in the last period, i.e.

$$\frac{W_T - W_0}{W_0}. \quad (5.3)$$

Obviously, other market indexes and portfolios could have been used in this comparison, but it was decided to include only cases where the same fifty assets are available for trading.

Table 5.1: Return on investment of the study cases.

| | Planned Strategy | Adjusted Strategy |
|---------------------------|------------------|-------------------|
| Model 1, $\phi = 0.05$ | 40.13% | 27.54% |
| Model 1, $\phi = 0.10$ | 71.87% | 64.61% |
| Model 1, $\phi = 0.15$ | 76.67% | 71.74% |
| Model 2, $\varphi = 0.90$ | 76.77% | 72.48% |
| Model 2, $\varphi = 0.95$ | 70.26% | 78.14% |
| Model 2, $\varphi = 1.00$ | 12.50% | 21.16% |
| IBrX-50 | 51.22% | 51.22% |
| Naive $1/n$ | 53.97% | 53.97% |

Chapter 6

Conclusions

This work addresses the multistage wealth allocation problem using conditional risk measures. Two formulations were proposed in a proper structure for the direct use of risk-neutral Stochastic Dual Dynamic Programming – SDDP. The first maximizes the expected value of the portfolio at the end of the investment and uses constraints to limit losses. The second minimizes the risk over the period and ensures a given profitability through constraints. In each model, two operation strategies were adopted: At first SDDP is used only once and plans decisions for all trading dates making simulations; the second runs SDDP on every trading date using always the newest information. Also, three risk/return thresholds were used in each model, representing different choices of an agent. This resulted in twelve cases that were numerically analyzed using real assets present in the IBrX-50 index.

The performance of the portfolios found by the optimization models was evaluated using stock prices collected during the period of the study. The return on investment was significantly higher than IBrX-50 index and a naive portfolio. The scenario tree was generated by the Geometric Brownian Motion – GBM. The great variety of available assets allowed a realistic representation of the problem. In addition, the combined use between GBM and SDDP was not found in other works. The computational effort was reasonable and it shows the relevance of the approach addressed here for dealing with this kind of problems. These are the main contributions of this work.

Only at one stage the scenario set failed to capture the pessimistic returns that actually occurred, reducing the portfolio value and making some of the thresholds to

be exceeded. Future works may use other techniques for scenario generation, especially those that can represent extreme events. Although they have a low probability of occurrence, it is especially useful to consider these events on solving risk-averse optimization problems. However, it is important to maintain the independence between the stages to use SDDP directly. Another approach is to generate extremely large trees and use scenario reduction techniques, in order to ensure that smaller trees be a good representation of reality.

An important direction to extend the research is the evaluation of time consistency between the planned policy and the one that is being adjusted with the SDDP reimplementation. Here this was not possible because the planned case was obtained by taking the average from the portfolios performing simulations of the future, and the performance of the portfolios was calculated using their real returns. Consistency was evaluated in each path and considering values from the objective function. In this context, it is interesting to investigate the effect of the risk/return constraints on the time consistency of the solutions, because the literature's models only optimize the risk or the convex combination between it and the expected value of the portfolio. In addition, this topic still is in discussion and new risk measures can be proposed, including non-linear models.

Finally, other analyses can be performed about the portfolios obtained here. In quantitative terms, scenario generation can be used for *ex post* evaluation, especially paths that were not considered by the optimization problem. Statistical measures commonly adopted in finance such as Sharpe ratio and Jensen's alpha can also be used. Qualitatively, these results can be discussed considering the economic and political environment of the country.

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Appendix A

Portfolio Composition

Table A.1: Portfolio composition of Model 1 with $\phi = 0.05$ (in %).

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| BRFS3 | 2.2 | 6.9 | 0 | 0 | 15.5 | 0 | 2.2 | 0 | 0 | 0 | 0 | 0 |
| BRKM5 | 4.0 | 0 | 0 | 3.7 | 0 | 0 | 4.0 | 0 | 0 | 0.9 | 0 | 0 |
| BRML3 | 0 | 14.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BVMF3 | 0 | 0 | 0 | 0 | 16.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CIEL3 | 0 | 0 | 0 | 27.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CPFE3 | 0 | 0 | 0 | 0 | 2.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CSAN3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.9 | 12.8 | 0 | 0 |
| CTIP3 | 0 | 0 | 1.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1 | 0 |
| EMBR3 | 0 | 6.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4.5 | 11.1 | 0 |
| EQTL3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6.1 | 0 |
| FIBR3 | 17.3 | 0 | 9.3 | 0 | 0.3 | 0 | 17.3 | 0 | 0 | 5.7 | 8.1 | 0 |
| GGBR4 | 29.3 | 0 | 0 | 0 | 0 | 0 | 29.3 | 0 | 0 | 7.9 | 0 | 0 |
| GOAU4 | 0 | 1.3 | 33.9 | 11.7 | 0 | 0 | 0 | 25.7 | 14.5 | 0.3 | 0 | 0 |
| HYPE3 | 0 | 0 | 0 | 0 | 2.7 | 21.5 | 0 | 0 | 0 | 7.9 | 0 | 2.3 |
| JBSS3 | 2.6 | 0 | 0 | 0 | 0 | 0 | 2.6 | 0 | 0 | 15.6 | 0 | 0 |
| KROT3 | 0 | 0 | 0 | 0 | 1.0 | 1.1 | 0 | 0 | 0 | 0 | 0 | 25.7 |
| LAME4 | 0 | 0 | 0 | 0 | 11.9 | 0 | 0 | 0 | 0 | 0 | 1.5 | 29.2 |
| LREN3 | 0 | 0 | 0 | 0 | 0.6 | 17.5 | 0 | 0 | 0 | 0 | 1.0 | 6.7 |
| MRVE3 | 13.0 | 0 | 0 | 0 | 0 | 0 | 13.0 | 0 | 0 | 0 | 0 | 0 |
| MULT3 | 0 | 0 | 0 | 0 | 0.6 | 0 | 0 | 0 | 0 | 0 | 28.8 | 0 |
| PCAR4 | 0 | 0 | 0.5 | 0 | 0 | 0 | 0 | 17.2 | 6.6 | 0 | 0 | 0 |
| PETR3 | 0 | 0 | 0 | 0 | 0 | 18.1 | 0 | 0 | 0 | 0 | 0 | 0 |
| PETR4 | 0 | 0 | 0 | 1.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| QUAL3 | 0 | 0 | 21.4 | 0 | 0 | 0 | 0 | 10.5 | 21.1 | 8.0 | 0 | 0 |
| RADL3 | 0 | 0 | 0 | 0 | 2.3 | 0 | 0 | 0 | 0 | 0 | 18.9 | 36.0 |
| SBSP3 | 0 | 0 | 0 | 0 | 0 | 11.4 | 0 | 0 | 0 | 0 | 0 | 0 |
| SMLE3 | 0 | 36.2 | 6.2 | 12.9 | 13.3 | 30.5 | 0 | 0 | 0 | 0 | 4.2 | 0 |
| SUZB5 | 3.8 | 0 | 13.1 | 6.8 | 32.9 | 0 | 3.8 | 6.1 | 21.0 | 13.8 | 0 | 0 |
| TIMP3 | 15.4 | 0 | 0 | 10.1 | 0 | 0 | 15.4 | 13.3 | 16.8 | 0 | 0 | 0 |
| USIM5 | 0 | 7.5 | 0 | 5.0 | 0 | 0 | 0 | 7.1 | 4.0 | 9.0 | 0 | 0 |
| VALE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 13.2 | 0 | 0 | 0 |
| VALE5 | 7.6 | 10.8 | 0 | 10.2 | 0 | 0 | 7.6 | 20.1 | 0 | 13.6 | 2.3 | 0 |
| VIVT4 | 0 | 0 | 0 | 10.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| WEGE3 | 4.8 | 15.6 | 14.3 | 0 | 0 | 0 | 4.8 | 0 | 0 | 0 | 17.9 | 0 |

Table A.2: Portfolio composition of Model 1 with $\phi = 0.10$ (in %).

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| BRAP4 | 0 | 0 | 12.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRFS3 | 0 | 0 | 0 | 0 | 16.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRKM5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10.2 | 0 | 0 |
| BRML3 | 0 | 10.6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BVMF3 | 0 | 0 | 0 | 0 | 5.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CCRO3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10.6 | 0 | 0 | 0 | 0 |
| CIEL3 | 0 | 0 | 0 | 3.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CSAN3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16.4 | 0 | 0 |
| CSNA3 | 25.8 | 0 | 0 | 0 | 0 | 0 | 25.8 | 0 | 0 | 0 | 0 | 0 |
| CTIP3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1 | 0 |
| EMBR3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11.1 | 0 |
| EQTL3 | 0 | 0 | 0 | 0 | 0.5 | 0 | 0 | 0 | 0 | 0 | 6.1 | 0 |
| ESTC3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5.4 | 0 | 0 | 0 | 0 |
| FIBR3 | 0 | 0 | 20.7 | 0 | 8.6 | 0 | 0 | 0 | 0 | 9.6 | 8.1 | 0 |
| GOAU4 | 28.0 | 28.7 | 40.1 | 34.3 | 0 | 0 | 28.0 | 49.9 | 10.1 | 14.6 | 0 | 0 |
| HYPE3 | 2.0 | 0 | 0 | 0 | 0 | 21.5 | 2.0 | 0 | 0 | 0 | 0 | 2.3 |
| JBSS3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8.6 | 0 | 0 |
| KROT3 | 0 | 0 | 0 | 0 | 1.1 | 1.1 | 0 | 0 | 0 | 0 | 0 | 25.7 |
| LAME4 | 0 | 0 | 0 | 0 | 3.0 | 0 | 0 | 0 | 0 | 0 | 1.5 | 29.2 |
| LREN3 | 0 | 0 | 0 | 0 | 1.9 | 17.5 | 0 | 0 | 0 | 0 | 1.0 | 6.7 |
| MULT3 | 0 | 0 | 0 | 0 | 4.5 | 0 | 0 | 0 | 0 | 0 | 28.8 | 0 |

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| NATU3 | 0 | 0 | 0 | 3.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| PCAR4 | 0 | 0 | 0 | 0 | 15.7 | 0 | 0 | 0 | 6.1 | 0 | 0 | 0 |
| PETR3 | 0 | 0 | 0 | 0 | 0 | 18.1 | 0 | 0 | 0 | 0 | 0 | 0 |
| QUAL3 | 14.5 | 0 | 0 | 0 | 0 | 0 | 14.5 | 0 | 34.5 | 24.3 | 0 | 0 |
| RADL3 | 0 | 0 | 0 | 0 | 11.8 | 0 | 0 | 0 | 0 | 0 | 18.9 | 36.0 |
| SBSP3 | 0 | 0 | 0 | 0 | 0 | 11.4 | 0 | 0 | 0 | 0 | 0 | 0 |
| SMLE3 | 0 | 40.4 | 0 | 0 | 6.8 | 30.5 | 0 | 0 | 0 | 0 | 4.2 | 0 |
| SUZB5 | 0 | 0 | 0 | 0 | 24.0 | 0 | 0 | 1.4 | 0 | 13.1 | 0 | 0 |
| TIMP3 | 5.6 | 0 | 0 | 28.2 | 0 | 0 | 5.6 | 11.2 | 9.4 | 0 | 0 | 0 |
| USIM5 | 5.5 | 12.9 | 0 | 19.1 | 0 | 0 | 5.5 | 0 | 4.2 | 3.1 | 0 | 0 |
| VALE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 35.9 | 0 | 0 | 0 |
| VALE5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 21.4 | 0 | 0 | 2.3 | 0 |
| VIVT4 | 0 | 0 | 26.5 | 11.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| WEGE3 | 18.5 | 7.4 | 0 | 0 | 0 | 0 | 18.5 | 0 | 0 | 0 | 17.9 | 0 |

Table A.3: Portfolio composition of Model 1 with $\phi = 0.15$ (in %).

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|------|------|------|------|--------------------------|-----|-----|------|------|-----|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| ABEV3 | 0 | 0 | 12.6 | 12.6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BBSE3 | 0 | 0 | 4.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRAP4 | 0 | 5.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRFS3 | 0 | 20.2 | 0 | 4.8 | 15.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRKM5 | 0 | 0 | 0 | 15.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRML3 | 0 | 5.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BVMF3 | 0 | 0 | 0 | 0 | 5.7 | 6.1 | 0 | 0 | 0 | 0 | 0 | 0 |
| CCRO3 | 0 | 0 | 4.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CIEL3 | 0 | 0 | 2.6 | 18.3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CPFE3 | 0 | 0 | 0.6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CSAN3 | 8.5 | 0 | 0 | 0 | 0 | 0 | 8.5 | 0 | 0 | 13.0 | 0 | 0 |
| CSNA3 | 13.6 | 0 | 0 | 0 | 0 | 0 | 13.6 | 8.5 | 0 | 0 | 0 | 0 |
| CTIP3 | 0 | 2.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1 | 0 |
| EGIE3 | 0 | 0 | 0 | 1.6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| EMBR3 | 0 | 0 | 0.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10.2 | 0 |
| EQTL3 | 0 | 0 | 12.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5.8 | 0 |
| ESTC3 | 0 | 0 | 0 | 4.4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FIBR3 | 0 | 0 | 0.9 | 0 | 10.5 | 19.8 | 0 | 0 | 0 | 4.8 | 8.0 | 0 |

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| GGBR4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16.4 | 0 | 0 |
| GOAU4 | 16.0 | 16.5 | 0 | 0 | 0 | 0 | 16.0 | 31.5 | 11.7 | 0 | 0 | 0 |
| HYPE3 | 0 | 0 | 0 | 0 | 0 | 24.3 | 0 | 0 | 0 | 8.5 | 0 | 2.6 |
| ITUB4 | 0 | 0 | 0 | 0 | 0 | 20.4 | 0 | 0 | 0 | 0 | 0 | 0 |
| JBSS3 | 0 | 2.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 15.4 | 0 | 0 |
| KROT3 | 0 | 0 | 0 | 0 | 0.8 | 3.1 | 0 | 0 | 0 | 0 | 0 | 27.1 |
| LAME4 | 0 | 0 | 0 | 0 | 3.2 | 0 | 0 | 0 | 0 | 0 | 1.5 | 26.0 |
| LREN3 | 0 | 0 | 0 | 0.1 | 2.1 | 5.6 | 0 | 0 | 0 | 0 | 0.9 | 6.4 |
| MULT3 | 0 | 14.7 | 0 | 0 | 7.5 | 0 | 0 | 0 | 0 | 1.5 | 29.4 | 0 |
| NATU3 | 0 | 0 | 0 | 3.2 | 0 | 0 | 0 | 0 | 16.1 | 0 | 0 | 0 |
| PCAR4 | 0 | 0 | 10.3 | 0 | 17.5 | 0 | 0 | 11.3 | 11.3 | 0 | 0 | 0 |
| QUAL3 | 16.6 | 0 | 0 | 0 | 0 | 0 | 16.6 | 6.3 | 19.4 | 10 | 0 | 0 |
| RADL3 | 0 | 0 | 0 | 0 | 12.1 | 0 | 0 | 0 | 0 | 0 | 19.7 | 37.9 |
| SMLE3 | 0 | 28.6 | 10 | 10.4 | 4.3 | 17.4 | 0 | 0 | 0 | 0 | 4.2 | 0 |
| SUZB5 | 0 | 0 | 14.3 | 6.3 | 19.8 | 3.4 | 0 | 0 | 0 | 11.3 | 0 | 0 |
| TIMP3 | 3.6 | 0 | 2.0 | 8.6 | 0 | 0 | 3.6 | 12.4 | 21.9 | 0 | 0 | 0 |
| USIM5 | 6.8 | 1.1 | 0 | 0 | 0 | 0 | 6.8 | 8.5 | 2.8 | 5.6 | 0 | 0 |
| VALE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 21.5 | 16.8 | 0 | 0 | 0 |
| VALE5 | 34.9 | 0 | 0 | 0 | 0 | 0 | 34.9 | 0 | 0 | 13.6 | 2.3 | 0 |
| VIVT4 | 0 | 0 | 21.4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| WEGE3 | 0 | 4.2 | 4.2 | 13.8 | 0.7 | 0 | 0 | 0 | 0 | 0 | 18.0 | 0 |

Table A.4: Portfolio composition of Model 2 with $\varphi = 0.90$ (in %).

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|-----|------|------|-----|--------------------------|------|-----|------|------|-----|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| BRFS3 | 0 | 0 | 0 | 0 | 19.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRKM5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5.7 | 3.4 | 0 |
| BRML3 | 0 | 22.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BVMF3 | 0 | 0 | 0 | 0 | 20.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CIEL3 | 0 | 0 | 0 | 12.3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CPFE3 | 0 | 0 | 0 | 0 | 1.9 | 0 | 0 | 0 | 0 | 0 | 6.4 | 0 |
| CSAN3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10.2 | 4.8 | 0 | 0 | 0 |
| CSNA3 | 32.0 | 0 | 0 | 0 | 0 | 0 | 32.0 | 5.3 | 0 | 0 | 0 | 0 |
| CTIP3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8.4 | 0 |
| EGIE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 18.4 | 0 | 0 |
| EMBR3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7.9 | 15.6 | 0 |

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|--------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| ESTC3 | 0 | 0 | 0 | 0.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FIBR3 | 0 | 0 | 14.3 | 0 | 0 | 0 | 0 | 0 | 0 | 22.8 | 16.2 | 12.6 |
| GGBR4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 26.1 | 14.0 | 0 | 0 | 0 |
| GOAU4 | 27.3 | 34.5 | 48.8 | 21.8 | 0 | 0 | 27.3 | 0 | 0 | 0 | 0 | 0 |
| HYPE3 | 11.0 | 0 | 0 | 0 | 2.7 | 21.5 | 11.0 | 0 | 0 | 0 | 0 | 0 |
| JBSS3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.6 | 15.6 | 18.0 | 5.4 | 0 |
| KLBN11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10.9 | 0 |
| KROT3 | 0 | 0 | 0 | 0 | 2.2 | 1.1 | 0 | 0 | 0 | 0 | 8.0 | 3.2 |
| LAME4 | 0 | 0 | 0 | 0 | 6.9 | 0 | 0 | 0 | 0 | 0 | 0 | 25.1 |
| LREN3 | 0 | 0 | 0 | 0 | 1.4 | 17.5 | 0 | 0 | 0 | 0 | 0 | 2.0 |
| MULT3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6.4 | 8.7 | 0 |
| NATU3 | 0 | 16.3 | 0 | 4.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| PCAR4 | 0 | 0 | 0 | 0 | 1.9 | 0 | 0 | 0 | 1.1 | 0 | 0 | 0 |
| PETR3 | 0 | 0 | 0 | 0 | 0 | 18.1 | 0 | 0 | 0 | 0 | 0 | 0 |
| QUAL3 | 0 | 0 | 5.9 | 0 | 0 | 0 | 0 | 19.7 | 28.2 | 7.9 | 0 | 0 |
| RADL3 | 0 | 0 | 0 | 0 | 4.2 | 0 | 0 | 0 | 0 | 0 | 7.1 | 56.3 |
| SBSP3 | 0 | 0 | 0 | 0 | 0 | 11.4 | 0 | 0 | 0 | 0 | 0 | 0 |
| SMLE3 | 0 | 14.7 | 0 | 3.4 | 5.6 | 30.5 | 0 | 0 | 0 | 0 | 4.9 | 0 |
| SUZB5 | 0 | 0 | 0 | 8.1 | 33.2 | 0 | 0 | 6.5 | 29.4 | 0 | 0 | 0.9 |
| TIMP3 | 8.7 | 0 | 9.3 | 26.2 | 0 | 0 | 8.7 | 0 | 0 | 0 | 0 | 0 |
| USIM5 | 3.0 | 11.6 | 0 | 16.3 | 0 | 0 | 3.0 | 0 | 0 | 0 | 0 | 0 |
| VALE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6.8 | 0 | 0 | 0 |
| VALE5 | 11.5 | 0 | 18.1 | 0 | 0 | 0 | 11.5 | 30.6 | 0 | 13.0 | 4.9 | 0 |
| VIVT4 | 0 | 0 | 3.5 | 6.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| WEGE3 | 6.6 | 0 | 0 | 0 | 0 | 0 | 6.6 | 0 | 0 | 0 | 0 | 0 |

Table A.5: Portfolio composition of Model 2 with $\varphi = 0.95$ (in %).

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|------|-----|-----|------|-----|--------------------------|-----|-----|-----|-----|-----|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| BRAP4 | 0 | 0 | 4.4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRFS3 | 0 | 10.9 | 0 | 0 | 23.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRKM5 | 0 | 0 | 0 | 8.2 | 0 | 0 | 0 | 0 | 2.4 | 7.4 | 0 | 0 |
| BRML3 | 0 | 7.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BVMF3 | 0 | 0 | 0 | 0 | 1.1 | 6.1 | 0 | 0 | 0 | 0 | 0 | 0 |
| CIEL3 | 0 | 0 | 0 | 2.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CPFE3 | 0 | 0 | 5.3 | 0 | 0 | 0 | 0 | 0 | 1.5 | 0 | 2.6 | 0 |

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|--------|-------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| CSAN3 | 26.6 | 0 | 0 | 0 | 0 | 0 | 26.6 | 22.0 | 3.3 | 0 | 0 | 0 |
| CTIP3 | 0 | 0 | 19.9 | 0 | 1.7 | 0 | 0 | 0 | 0 | 7.0 | 12.0 | 0 |
| EGIE3 | 0 | 0 | 0 | 7.2 | 0 | 0 | 0 | 0 | 8.5 | 5.3 | 0 | 0 |
| EMBR3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9.2 | 6.5 | 13.2 | 0 |
| EQTL3 | 0 | 0 | 1.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ESTC3 | 0 | 0 | 0 | 2.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FIBR3 | 0 | 0 | 11.6 | 0 | 0.1 | 19.7 | 0 | 15.9 | 13.3 | 14.6 | 13.4 | 9.4 |
| GGBR4 | 0 | 0 | 6.5 | 0 | 0 | 0 | 0 | 0 | 6.2 | 4.9 | 5.6 | 0 |
| GOAU4 | 35.8 | 0.6 | 0.3 | 0 | 0 | 0 | 35.8 | 12.3 | 0 | 0 | 0 | 0 |
| HYPE3 | 0 | 0 | 0 | 0 | 0.8 | 24.3 | 0 | 0 | 2.8 | 0 | 0 | 0 |
| ITUB4 | 0 | 0 | 0 | 0 | 0 | 20.4 | 0 | 0 | 0 | 0 | 0 | 0 |
| JBSS3 | 0 | 21.2 | 0 | 0.2 | 0 | 0 | 0 | 2.7 | 9.3 | 19.8 | 3.9 | 0 |
| KLBN11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17.4 | 0 |
| KROT3 | 0 | 0 | 0 | 0 | 0.4 | 3.1 | 0 | 0 | 0 | 0 | 9.1 | 3.1 |
| LAME4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 29.2 |
| LREN3 | 0 | 0 | 0 | 0 | 1.6 | 5.6 | 0 | 0 | 0 | 0 | 0 | 2.1 |
| MULT3 | 0 | 0 | 0 | 0 | 10.5 | 0 | 0 | 0 | 7.8 | 13.4 | 4.0 | 0 |
| NATU3 | 0 | 20.2 | 0 | 10.3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| PCAR4 | 0 | 0 | 0.9 | 0 | 13.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| PETR4 | 0 | 0 | 0 | 4.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| QUAL3 | 0 | 0 | 3.9 | 0 | 0 | 0 | 0 | 19.7 | 10.9 | 1.0 | 0 | 0 |
| RADL3 | 0 | 3.9 | 0 | 0 | 4.0 | 0 | 0 | 0 | 0 | 14.9 | 6.4 | 55.4 |
| SBSP3 | 0 | 0 | 0.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SMLE3 | 0 | 13.7 | 12.5 | 9.0 | 7.4 | 17.4 | 0 | 0 | 0 | 0 | 9.2 | 0 |
| SUZB5 | 0 | 0 | 8.8 | 35.5 | 34.9 | 3.4 | 0 | 16.7 | 11.1 | 0 | 0 | 0.7 |
| TIMP3 | 6.9 | 0 | 6.7 | 6.3 | 0 | 0 | 6.9 | 0 | 0 | 0 | 0 | 0 |
| VALE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3.8 | 13.8 | 0 | 0 | 0 |
| VALE5 | 30.7 | 0 | 0 | 0 | 0 | 0 | 30.7 | 6.9 | 0 | 0.3 | 3.1 | 0 |
| VIVT4 | 0 | 0 | 11.2 | 13.6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| WEGE3 | 0 | 21.8 | 5.4 | 0 | 0 | 0 | 0 | 0 | 0 | 4.9 | 0 | 0 |

Table A.6: Portfolio composition of Model 2 with $\varphi = 1.00$ (in %).

| | Planned Strategy (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|-------------------------|-----|-----|-----|------|-----|--------------------------|-----|-----|-----|-----|-----|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| ABEV3 | 0 | 0 | 0 | 0 | 29.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRAP4 | 0 | 3.3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | Planned Strategy (2016) (2016) | | | | | | Adjusted Strategy (2016) | | | | | |
|-------|--------------------------------|------|------|------|------|------|--------------------------|------|------|------|------|------|
| | Fev | Apr | Jun | Aug | Oct | Dec | Fev | Apr | Jun | Aug | Oct | Dec |
| BRFS3 | 0 | 15.4 | 7.3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BRKM5 | 0 | 0 | 4.8 | 7.5 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 0 |
| BVMF3 | 0 | 0 | 0 | 0 | 0 | 13.9 | 0 | 0 | 0 | 0 | 0 | 0 |
| CIEL3 | 0 | 0 | 0 | 0 | 0 | 6.9 | 0 | 0 | 0 | 0 | 0 | 0 |
| CPFE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24.5 | 3.4 | 0 |
| CSAN3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 18.1 | 1.9 | 0 | 0 |
| CTIP3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.3 | 0 |
| EMBR3 | 0 | 0 | 31.8 | 6.7 | 0 | 0 | 0 | 0 | 0 | 4.0 | 11.3 | 0 |
| EQTL3 | 0 | 0 | 0 | 29.7 | 21.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FIBR3 | 3.3 | 0 | 0 | 3.2 | 0 | 0 | 3.3 | 0 | 0 | 0.6 | 11.9 | 0 |
| GGBR4 | 1.8 | 0 | 0 | 0 | 0 | 0 | 1.8 | 24.2 | 0 | 0 | 0 | 0 |
| GOAU4 | 18.4 | 29.1 | 0 | 0 | 0 | 0 | 18.4 | 2.5 | 11.2 | 0 | 0 | 0 |
| HYPE3 | 0 | 0 | 0 | 0 | 0 | 51.6 | 0 | 0 | 0 | 4.5 | 0 | 2.0 |
| JBSS3 | 5.3 | 6.3 | 6.1 | 0.4 | 0 | 0 | 5.3 | 7.8 | 15.2 | 16.8 | 0 | 0 |
| KROT3 | 0 | 1.9 | 0 | 0 | 0.6 | 3.5 | 0 | 0 | 0 | 0 | 2.4 | 24.3 |
| LAME4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32.6 |
| LREN3 | 0 | 0 | 0 | 4.4 | 1.3 | 11.2 | 0 | 0 | 0 | 0 | 0.2 | 7.1 |
| MULT3 | 0 | 0 | 25.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1 | 17.4 | 0 |
| PCAR4 | 19.3 | 0 | 0 | 0 | 0 | 0 | 19.3 | 20.3 | 7.8 | 2.0 | 0 | 0 |
| PETR4 | 0 | 5.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| QUAL3 | 7.4 | 29.8 | 0 | 4.7 | 0 | 0 | 7.4 | 0 | 7.8 | 3.9 | 0 | 0 |
| RADL3 | 0 | 0 | 24.3 | 37.1 | 0.5 | 0 | 0 | 0 | 0 | 0 | 28.2 | 34.0 |
| SMLE3 | 0 | 8.6 | 0 | 0 | 12.9 | 0 | 0 | 0 | 0 | 0 | 10.7 | 0 |
| SUZB5 | 0 | 0 | 0 | 1.6 | 8.3 | 13.0 | 0 | 3.2 | 23.7 | 27.0 | 0 | 0 |
| TIMP3 | 3.9 | 0 | 0 | 0 | 0 | 0 | 3.9 | 16.0 | 2.3 | 0 | 0 | 0 |
| USIM5 | 5.5 | 0 | 0 | 0 | 0 | 0 | 5.5 | 5.7 | 3.7 | 0 | 0 | 0 |
| VALE3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.3 | 6.7 | 14.8 | 0 | 0 |
| VIVT4 | 0 | 0 | 0 | 0 | 25.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| WEGE3 | 35.1 | 0 | 0 | 4.8 | 0 | 0 | 35.1 | 19.0 | 3.6 | 0 | 2.0 | 0 |